Problem 2

Solution.

We can use a simple array A to be our underlying data structure. Array A contains a
set S of integers. The operations are as follows.

INSERT(S,x): The element can be inserted to the last element in A.

DELETE-LARGER-HALF(S): First, find the median mid of S using array A. We then
partition S into two subsets, S_1 and S_2, where all the elements in S_1 are less than mid and
the elements in S_2 are larger than and equal to mid by comparing each element in A with
mid. Last, we delete S_2 and set S = S_1.

Suppose there are m operations performed on S. For each i = 1, 2, . . . , n, let c_i be the
actual cost of the i-th operation and A_i be the data structure that results after applying the
i-th operation to data structure A_{i-1}. Since we use the linear time algorithm for finding
the median, the running time of the algorithm for discarding the larger half is O(m) for
an m-element array. Suppose the actual time for discarding the larger half is km for some
constant k. We define the potential function Φ on each data structure A_i to be

Φ(A_i) = 2k × |A_i|

Where A_i is the number of elements in array A_i. For an operation i, we analyze the
cost by considering the following two cases.

INSERT(S,x): The actual cost c_i for insertion is 1. Then, the amortized cost

\[ \bar{c} = c_i + \Phi(A_i) - \Phi(A_{i-1}) \]
\[ = 1 + 2k \times |A_i| - 2k \times |A_{i-1}| \]
\[ = 1 + 2k \]

since |A_i| = |A_{i-1}| + 1 when inserting an element to A_{i-1}. The amortized cost therefore
is O(1).

DELETE-LARGER-HALF(S): As mentioned above, the actual cost for deleting the
larger half is km if there are m elements. Then, the amortized cost

\[ c_i = c_i + \Phi(A_i) - \Phi(A_{i-1}) \]
\[ = k \times |A_{i-1}| + 2k \times |A_i| - 2k |A_{i-1}| \]
\[ = k \times |A_{i-1}| + 2k \times \frac{1}{2} |A_{i-1}| - 2k |A_{i-1}| \]
\[ = k \times |A_{i-1}| + k \times |A_{i-1}| - 2k \times |A_{i-1}| \]
\[ = 0 \]

since |A_i| = \frac{1}{2} |A_{i-1}| when deleting the larger half of A_{i-1}. The amortized cost therefore
is 0.

Therefore, the m operations run in O(m) time. ■