

Rounding Errors and Volatility Estimation ^{*}

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Abstract

Financial prices are often discretized - to the nearest cent, for example. Thus we can say prices are observed with rounding error. Rounding errors affect the estimation of volatility and understanding them becomes important especially when we use high frequency data. We study the asymptotic behavior of the Realized Volatility (RV) which is commonly used as an estimator of the integrated volatility. We prove the convergence of the RV and scaled RV under different conditions on the rounding level and the number of observations. A bias-corrected volatility estimator is proposed and an associated central limit theorem is shown. Simulation results show that improvement in statistical properties can be substantial.

Keywords: Rounding Errors, Bias-correction, Diffusion Process, Market Microstructure, Realized Volatility

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1 Introduction

We consider a security price process S , whose logarithm $X = \log S$ follows

$$dX_t = \mu_t dt + \sigma_t dW_t. \quad (1.1)$$

In other words, S is the solution to the stochastic differential equation

$$dS_t = (\mu_t + \frac{1}{2}\sigma_t^2)S_t dt + \sigma_t S_t dW_t, \quad t \in [0, 1] \quad (1.2)$$

where W_t is a standard Brownian motion. We assume that μ_t and σ_t are continuous random processes satisfying some regularity conditions specified in section 2.1.

It is a common practice in finance to use the sum of frequently sampled squared returns (which is called the Realized Volatility (RV)) to estimate the integrated volatility $\int_0^1 \sigma_t^2 dt$. However, recent empirical studies in finance have showed evidence that market microstructure makes this estimator fail when the prices are sampled at high frequencies, and sampling sparsely gives more reasonable estimates. We investigate the case when the contamination due to market microstructure is simply round-off errors.

More specifically, let α_n be a sequence of positive numbers which represents the accuracy of measurement when one observes the process n times during the time period $[0,1]$. Suppose at time i/n ($i = 0, \dots, n$), one observes the value $k\alpha_n$ when the true value $S_{i/n}$ is in $[k\alpha_n, (k+1)\alpha_n)$ with $k \in \mathbb{Z}$. For every real s we denote by $s^{(\alpha_n)} = \alpha_n \lfloor s/\alpha_n \rfloor$ its rounded-off value at level α_n . We investigate the asymptotic behavior of the RV

$$V^n = \sum_{i=1}^n (\log(S_{i/n}^{(\alpha_n)}) - \log(S_{(i-1)/n}^{(\alpha_n)}))^2. \quad (1.3)$$

Jacod (1996) and Delattre and Jacod (1997) have previously studied the problem of inference for volatility based on a rounded Brownian motion. While this work is the inspiration for our work, these earlier results are not quite as relevant to securities prices, as rounding happens on the original (dollar, euro, etc) scale and not on the log

scale. As we shall see later in this paper, the more realistic type of rounding leads to a bias which is no longer nonrandom (as in section 4 of Delattre and Jacod (1997)), but instead requires a somewhat more complicated correction. Rounding has also been studied by Zeng (2003), who has developed a Bayesian inference algorithm for this problem; and more recently by Rosenbaum (2009), who has proposed an alternative estimation approach using absolute values of the increments. Models involving both rounding and additive error are studied in Li and Mykland (2007) and Jacod, Li, Mykland, Podolskij, and Vetter (2009), among others.

We shall prove the convergence of V^n . This shows what the problems of using the RV are, when the rounding errors are present; and explains why “sampling sparsely” could be a practically helpful way to estimate the volatility (however, “sampling sparsely” doesn’t solve all the problems). We then propose a bias-corrected estimator, and prove an associated central limit theorem. Simulation results show that our bias-corrected estimator can give substantial improvement in statistical accuracy. Our main bias correction applies to the case of “small rounding”, as in Delattre and Jacod (1997). This kind of asymptotics is quite realistic in practice, cf. the findings for additive error in Zhang, Mykland, and Aït-Sahalia (2009). Small rounding asymptotics has also been studied in Kolassa and McCullagh (1990), where it is shown to be related to additive error. We also discuss what happens to the RV when the rounding is not “small”.

These main results are presented in Section 2. Section 3 is devoted to the proof, and Section 4 to conclusions and discussion.

2 THE MAIN RESULTS

2.1 The Theorems

We assume that the latent security price process S_t follows (1.2), where σ_t is a (non-random) function of S_t , of class \mathcal{C}^5 on $[0, \infty)$ (In the Black-Scholes model, $\sigma_t \equiv \sigma$ is a constant). Assume further that μ_t is a continuous random process (in particular, it is locally bounded).

Let $\beta_n = \alpha_n \sqrt{n}$.

Theorem 1. *Under the condition that $\alpha_n \rightarrow 0$ as $n \rightarrow \infty$ in such a way that $\beta_n \rightarrow \beta \in [0, \infty)$, we have*

$$V^n \rightarrow_P \int_0^1 \sigma_t^2 dt + \frac{\beta^2}{6} \int_0^1 \frac{1}{S_t^2} dt - \frac{\beta^2}{\pi^2} \int_0^1 \frac{1}{S_t^2} \sum_{k=1}^{\infty} \frac{1}{k^2} \exp \left\{ -2\pi^2 k^2 \frac{\sigma_t^2 S_t^2}{\beta^2} \right\} dt.$$

One sees from this result that the bias is increasing in β , and is always positive when $\beta \neq 0$. It blows up to ∞ as β approaches ∞ . Also, the bias is smaller when S_t , $t \in [0, 1]$ is bigger. These are consistent with the empirical evidence that

- 1) subsampling helps (when one replace the n with a smaller sample size, the limiting β will be smaller, hence will result in a smaller bias); and
- 2) the rounding effect is less serious for more expensive stocks (the bias is smaller for higher values of S_t).

Theorem 1 shows that when $\beta_n \rightarrow 0$, one can have the consistency of V^n . The following theorem tells us about the convergence rate.

Theorem 2. *Under the condition that $\beta_n = O(n^{-\gamma})$ for some $\gamma > 0$, we have*

$$\sqrt{n} \left(V^n - \int_0^1 \sigma_t^2 dt - \frac{\beta_n^2}{6} \int_0^1 \frac{1}{S_t^2} dt \right) \rightarrow_{\mathcal{L}\text{-stably}} \int_0^1 \sqrt{2} \sigma_t^2 dB_t,$$

where B is a standard Brownian Motion independent of W .

In this case, there is still a finite sample bias of size $\frac{\beta_n^2}{6} \int_0^1 \frac{1}{S_t^2} dt$. One can estimate the bias and find a bias-corrected estimator as the following.

Theorem 3. *Assume that $\beta_n = O(n^{-\gamma})$ for some $\gamma > 0$, and let*

$$V_0^n := V^n - \frac{\alpha_n^2}{6} \sum_{i=1}^n \frac{1}{(S_{i/n}^{(\alpha_n)})^2}.$$

Then as $n \rightarrow \infty$,

$$\sqrt{n} \left(V_0^n - \int_0^1 \sigma_t^2 dt \right) \rightarrow_{\mathcal{L}\text{-stably}} \int_0^1 \sqrt{2} \sigma_t^2 dB_t,$$

where B is a standard Brownian Motion independent of W .

One can see from the simulation results in the next section that this bias-correction can lead to substantially improved confidence intervals.

Remark 1. One may find another quantity natural to study:

$$\tilde{V}^n = \sum_{i=1}^n \frac{(S_{i/n}^{(\alpha_n)} - S_{(i-1)/n}^{(\alpha_n)})^2}{S_{(i-1)/n}^{(\alpha_n)}}.$$

Conclusions about this quantity \tilde{V}^n can be easily drawn based on the proofs in section 3 of the above theorems. In particular, we can know that when $\beta \in [0, \infty)$, \tilde{V}^n has the same limit in probability as V^n , hence the same asymptotic bias. For the case when $\beta_n = O(n^{-\gamma})$ for some $\gamma > 0$, \tilde{V}^n needs the same bias correction term as V^n .

Remark 2. The condition of small rounding ($\alpha_n \rightarrow 0$) guarantees us to have the asymptotic results as above. In practice, we make use of these asymptotic results via expansion – we observe only one α_n and one n for a particular price process. If we believe we can rely on the asymptotic theory of small rounding, then we can make a correction as in Theorem 3, and have a better estimator (please refer to the simulation studies for further illustration of the use of these results).

Under the condition that the rounding level does not shrink with n but stays fixed, we have the following result.

Theorem 4. *Let the accuracy of measurement $\alpha_n \equiv \alpha$ be independent on the number of observations n . Consider the case when $\sigma_t \equiv \sigma$ for $t \in [0, 1]$. Redefine $S_{i/n}^{(\alpha)} = \alpha$ if*

$S_{i/n}^{(\alpha)} = 0$. As $n \rightarrow \infty$,

$$\frac{1}{\sqrt{n}}V^n \rightarrow_P \frac{1}{\sigma} \sqrt{\frac{2}{\pi}} \sum_{k=1}^{\infty} L_1^{\log((k+1)\alpha)} \left(\log \frac{k+1}{k}\right)^2,$$

where L_t^a is the local time at the level a of the continuous semimartingale $X_t = \log S_t$ (see Revuz and Yor (1999), page 222).

Note that by redefining $S_{i/n}^{(\alpha)} = \alpha$ if $S_{i/n}^{(\alpha)} = 0$, we rule out the possibility of taking the logarithm of zero when calculating the Realized Volatility. In practice, this simply means that the security price never go below the smallest rounding grid (1 cent if $\alpha = 0.01$) during the time interval that we consider; which is usually the case.

One sees that the Realized Volatility V^n blows up as the sample frequency n becomes higher (with rate \sqrt{n}), just as in Jacod (1996), though the form of the limit is different.

2.2 The Simulation Results

Consider the case when $\sigma_t \equiv \sigma$ for $t \in [0, 1]$. Denote by V^n_CI and $V_0^n_CI$ the nominal 95% confidence interval (CI) based on V^n and V_0^n respectively, as follows.

The naive CI based on V^n relies on the classical theory of the Realized Volatility, which says that when there is no observation error,

$$\sqrt{n}[V^n - \sigma^2] \rightarrow_{\mathcal{L}} N(0, 2\sigma^4).$$

The resulting nominal 95% CI is

$$V^n_CI = \left[V^n - 1.96 * \sqrt{2(V^n)^2/n}, V^n + 1.96 * \sqrt{2(V^n)^2/n} \right].$$

Our findings above indicate that there are problems with using the classical theory of the Realized Volatility when the rounding errors are present. And we have found

a simple bias-corrected estimator which should work when $\alpha_n\sqrt{n}$ is reasonably small

$$V_0^n = V^n - \frac{\alpha_n^2}{6} \sum_{i=1}^n \frac{1}{(S_{i/n}^{(\alpha_n)})^2}.$$

By Theorem 3,

$$\sqrt{n}[V_0^n - \sigma^2] \rightarrow_{\mathcal{L}} N(0, 2\sigma^4).$$

Our adjusted nominal 95% *CI* is then

$$V_0^n\text{-}CI = \left[V_0^n - 1.96 * \sqrt{2(V_0^n)^2/n}, V_0^n + 1.96 * \sqrt{2(V_0^n)^2/n} \right].$$

To examine the performance of these volatility estimators V^n and V_0^n , we did the following simulation.

We simulated sample paths from (1.2) with $\mu = 0$, $\sigma = 0.1$. We ran 10000 simulations of a single one-day period. The starting price of the day was taken to be $S_0 = 6$. For rounding, we used a fixed rounding level $\alpha_n \equiv \alpha = 0.01$, to be consistent with the real world where the stock prices are often rounded to cent.

Table 1 shows the simulation results. The first column of the table gives the sample size; the second column gives the corresponding sample frequencies; and the third column gives the pre-limiting β_n (so we see how our small rounding asymptotic theory works for the case of finite sample size and fixed rounding level). The last two columns contain three items each. The notation “*f*” stands for “actual coverage frequency”, it records the frequency with which the true parameter fell in the confidence intervals based on the corresponding volatility estimators V^n and V_0^n ; “*l*” stands for “average length of the confidence interval”, so we get some idea of how wide the confidence intervals are; “*b*” stands for “finite sample bias”, so we see how much and to which direction the estimators are biased.

Comparing V^n to V_0^n , we see that when the sample frequency is relatively low (5 min - 1 min; see 2nd ~ 3rd row), both V^n and V_0^n perform well in the sense that their nominal 95% confidence intervals are doing their jobs – these actual coverage frequencies are about 95%. This is consistent with the empirical evidence that sub-

samp. size	samp. freq.	' β ' $\alpha\sqrt{n}$		V^n_CI	$V_0^n_CI$
78	5 min	0.088	f:	93.12%	92.82%
			l:	$6.22 * 10^{-3}$	$6.19 * 10^{-3}$
			b:	$-9.44 * 10^{-5}$	$-1.31 * 10^{-4}$
234	1 min	0.153	f:	95.33%	94.49%
			l:	$3.65 * 10^{-3}$	$3.61 * 10^{-3}$
			b:	$6.97 * 10^{-5}$	$-3.98 * 10^{-5}$
1170	20 sec	0.342	f:	78.05%	93.34%
			l:	$1.71 * 10^{-3}$	$1.62 * 10^{-3}$
			b:	$5.39 * 10^{-4}$	$-9.03 * 10^{-6}$
2340	10 sec	0.484	f:	7.83%	92.52%
			l:	$1.27 * 10^{-3}$	$1.15 * 10^{-3}$
			b:	$1.09 * 10^{-3}$	$-3.79 * 10^{-6}$
4680	5 sec	0.697	f:	0	89.07%
			l:	$9.88 * 10^{-4}$	$8.10 * 10^{-4}$
			b:	$2.19 * 10^{-3}$	$-2.28 * 10^{-6}$
23400	1 sec	1.53	f:	0	25.05%
			l:	$7.46 * 10^{-4}$	$3.49 * 10^{-4}$
			b:	$1.06 * 10^{-2}$	$-3.61 * 10^{-4}$

Table 1: Performance of the nominal 95% confidence intervals based on V^n and V_0^n . “f”: actual coverage frequency of the confidence intervals; “l”: average length of the confidence intervals; “b”: finite sample bias.

sampling helps. But since the convergence rate is square root of n, the confidence intervals are wide when the n is small. Going down to the 4th ~ 6th row, we see that when the sample frequency goes a bit higher (20 sec - 5 sec), the problems with the Realized Volatility show up, the coverage frequency goes down from about 95% to 0; while the $V_0^n_CI$ still perform quite well. Also from the biases we see that the Realized Volatility goes to something much bigger than the true value, while the V_0^n stays close to the true parameter. So we see that overall, V_0^n does a better job than the Realized Volatility V^n .

Note that for very high-frequency (1 sec, 7th row), the bias-corrected volatility estimator doesn't perform as well either. This is as expected, since the bias-corrected estimator is built upon the asymptotic theory that requires the condition $\alpha_n\sqrt{n} \rightarrow 0$,

which is never true in practice. For a fixed rounding level, if the sample frequency goes higher and higher, our bias correction would eventually fail. The failure at really high frequency would probably happen to all other RV-based volatility estimators, too, as a direct consequence of the Theorem 4 (see Theorem 2 in Li and Mykland (2007) for a result of another RV-based volatility estimator). The above simulation suggests that for the given price level and the rounding level, when the sample frequency is lower than 5 seconds, our bias correction can be very helpful.

3 PROOFS OF THE MAIN RESULTS

3.1 Preparation

We assume without loss of generality (see section 3.4 for further justification) that $\mu_t = 0$, in which case

$$d \log S_t = \sigma_t dW_t; \quad (3.1)$$

and that there exist nonrandom constants $L_\sigma, U_\sigma \in (0, \infty)$, such that

$$L_\sigma \leq \sigma_t \leq U_\sigma \quad \text{for } t \in [0, 1].$$

More Notation:

$$\begin{aligned} A_m &:= \{\omega \in \Omega : S_t(\omega)_{t \in [0,1]} \in [\frac{1}{m}, m]\}; \\ B_n &:= \{\omega \in \Omega : \max_{1 \leq i \leq n} \sqrt{n} \left| \frac{S_{i/n}}{S_{(i-1)/n}} - 1 \right| \leq 2 \log n\}; \\ Y_{i,n} &:= \sqrt{n}(S_{i/n}^{(\alpha_n)} - S_{(i-1)/n}^{(\alpha_n)}); \\ U(n, \phi) &= \frac{1}{n} \sum_{i=1}^n \phi(S_{(i-1)/n}^{(\alpha_n)}, Y_{i,n}) \quad \text{for } \phi : \mathbb{R}^2 \rightarrow \mathbb{R}; \end{aligned} \quad (3.2)$$

$h(\cdot)$: density of the standard normal law ;

$h_s(\cdot)$: density of the normal law $N(0, s^2)$.

Lemma 1. $P(B_n) \rightarrow 1$ as $n \rightarrow \infty$.

Proof: By (3.1),

$$S_{i/n}/S_{(i-1)/n} = \exp\left\{\int_{(i-1)/n}^{i/n} \sigma_s dW_s\right\}.$$

Note that for any $i = 1, 2, \dots, n$,

$$\begin{aligned} & E\left(\exp\left\{\sqrt{n} \int_{(i-1)/n}^{i/n} \sigma_s dW_s\right\}\right) \\ & \leq E\left(\exp\left\{\sqrt{n} \int_{(i-1)/n}^{i/n} \sigma_s dW_s - \frac{1}{2}n \int_{(i-1)/n}^{i/n} \sigma_s^2 ds + \frac{1}{2}U_\sigma^2\right\}\right) \\ & = \exp\left\{\frac{1}{2}U_\sigma^2\right\}. \end{aligned}$$

Hence for any $a > 0$

$$\begin{aligned} & P\left(\int_{(i-1)/n}^{i/n} \sigma_s dW_s > a\right) \\ & = P\left(\exp\left\{\sqrt{n} \int_{(i-1)/n}^{i/n} \sigma_s dW_s\right\} > \exp\{\sqrt{na}\}\right) \\ & \leq \frac{\exp\left\{\frac{1}{2}U_\sigma^2\right\}}{\exp\{\sqrt{na}\}}. \end{aligned}$$

Therefore,

$$\begin{aligned} & P\left(\max_{1 \leq i \leq n} \left(\sqrt{n} \left(\frac{S_{i/n}}{S_{(i-1)/n}} - 1\right)\right) > 2 \log n\right) \\ & = P\left(\max_{1 \leq i \leq n} \left(\frac{S_{i/n}}{S_{(i-1)/n}}\right) > \frac{2 \log n}{\sqrt{n}} + 1\right) \\ & = P\left(\max_{1 \leq i \leq n} \left(\int_{(i-1)/n}^{i/n} \sigma_s dW_s\right) > \log\left(\frac{2 \log n}{\sqrt{n}} + 1\right)\right) \\ & \leq n \frac{\exp\left\{\frac{1}{2}U_\sigma^2\right\}}{\exp\left\{\sqrt{n} \left(\log\left(\frac{2 \log n}{\sqrt{n}} + 1\right)\right)\right\}} \\ & \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

A parallel argument gives the conclusion that

$$P \left(\max_{1 \leq i \leq n} \left(\sqrt{n} \left(1 - \frac{S_{i/n}}{S_{(i-1)/n}} \right) \right) > 2 \log n \right) \rightarrow 0 \text{ as } n \rightarrow \infty,$$

hence the conclusion.

Lemma 2. *If $\sqrt{n}\alpha_n \rightarrow \beta \in [0, \infty)$, then for any m , there exist N large and $c_m \in (0, \frac{1}{m}]$ such that for all $n \geq N$, $i = 0, 1, 2, \dots, n$,*

$$S_{i/n}^{(\alpha_n)} \geq c_m \text{ on } A_m.$$

Proof:

$$\forall i = 0, 1, 2, \dots, n, \quad S_{i/n}^{(\alpha_n)} \geq S_{i/n} - \alpha_n;$$

and

$$S_{i/n} \geq \frac{1}{m} \text{ on } A_m, \text{ and } \alpha_n \rightarrow 0 \text{ as } n \rightarrow \infty,$$

hence the conclusion.

Lemma 3. *Suppose that $\beta_n = \sqrt{n}\alpha_n \rightarrow \beta \in [0, \infty)$, then for any fixed $m > 0$,*

$$\sup_{\omega \in A_m \cap B_n} \frac{Y_{i,n}}{\sqrt{n} S_{(i-1)/n}^{(\alpha_n)}} = O \left(\frac{\log n}{\sqrt{n}} \right).$$

Proof: On $A_m \cap B_n$,

$$|Y_{i,n}| = \sqrt{n} |S_{i/n}^{(\alpha_n)} - S_{(i-1)/n}^{(\alpha_n)}| \leq \sqrt{n} (|S_{i/n} - S_{(i-1)/n}| + 2\alpha_n) \leq 2m \log n + 2\beta_n.$$

By lemma 2, one can find a $c_m \in (0, \frac{1}{m}]$ such that for large n , on $A_m \cap B_n$,

$$\frac{|Y_{i,n}|}{\sqrt{n} S_{(i-1)/n}^{(\alpha_n)}} \leq \frac{2m \log n + 2\beta_n}{\sqrt{n} c_m}.$$

Since $\beta_n \rightarrow \beta < \infty$, the above inequality implies that $\sup_{\omega \in A_m \cap B_n} \frac{Y_{i,n}}{\sqrt{n} S_{(i-1)/n}^{(\alpha_n)}}$ is $O \left(\frac{\log n}{\sqrt{n}} \right)$.

Lemma 4. Let $\beta > 0$, then for all $\sigma, x > 0$,

$$\int_0^1 \int h(y) \left(\frac{\beta \lfloor u + y\sigma x / \beta \rfloor}{x} \right)^2 dy du = \sigma^2 + \frac{1}{x^2} \left(\frac{\beta^2}{6} - \frac{\beta^2}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{k^2} \exp\left\{-2\pi^2 k^2 \frac{\sigma^2 x^2}{\beta^2}\right\} \right).$$

Proof:

$$\begin{aligned} & \int_0^1 \int h(y) \left(\frac{\beta \lfloor u + y\sigma x / \beta \rfloor}{x} \right)^2 dy du \\ &= E \left(\left(\frac{\beta \lfloor U + Y\sigma x / \beta \rfloor}{x} \right)^2 \right), \quad U \sim \text{unif}[0, 1], \quad Y \sim N(0, 1) \\ &= \frac{\beta^2}{x^2} E(\lfloor U + Y\sigma x / \beta \rfloor)^2 \\ &= \frac{\beta^2}{x^2} E(\lfloor U + Z \rfloor)^2, \quad Z \sim N(0, \frac{\sigma^2 x^2}{\beta^2}) \\ &= \frac{\beta^2}{x^2} E(E(\lfloor U + Z \rfloor^2 | Z)) \\ &= \frac{\beta^2}{x^2} E((Z - \{Z\})^2(1 - \{Z\}) + (Z + 1 - \{Z\})^2\{Z\}) \\ &= \frac{\beta^2}{x^2} (EZ^2 + E(\{Z\}(1 - \{Z\}))) \\ &= \sigma^2 + \frac{1}{x^2} \left(\frac{\beta^2}{6} - \frac{\beta^2}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{k^2} \exp\left\{-2\pi^2 k^2 \frac{\sigma^2 x^2}{\beta^2}\right\} \right), \end{aligned}$$

where $\{z\} = z - \lfloor z \rfloor$ is the fractional part of z .

The last equality above is proved by using the Fourier expansion for function $f(z) = \{z\} - \{z\}^2$.

3.2 Proof of Theorem 1

Recall that V^n is defined in (1.3). For large n ,

$$\begin{aligned}
& V^n I_{A_m \cap B_n} \\
&= \sum_{i=1}^n (\log S_{i/n}^{(\alpha_n)} - \log S_{(i-1)/n}^{(\alpha_n)})^2 I_{A_m \cap B_n} \\
&= \frac{1}{n} \sum_{i=1}^n \left[\sqrt{n} \log \left(\frac{S_{i/n}^{(\alpha_n)} - S_{(i-1)/n}^{(\alpha_n)}}{S_{(i-1)/n}^{(\alpha_n)}} + 1 \right) \right]^2 I_{A_m \cap B_n} \\
&= \frac{1}{n} \sum_{i=1}^n \left[\sqrt{n} \log \left(\frac{Y_{i,n}}{\sqrt{n} S_{(i-1)/n}^{(\alpha_n)}} + 1 \right) \right]^2 I_{A_m \cap B_n} \\
&= \frac{1}{n} \sum_{i=1}^n \left[\sqrt{n} \left(\frac{Y_{i,n}}{\sqrt{n} S_{(i-1)/n}^{(\alpha_n)}} - \frac{1}{2} \left(\frac{Y_{i,n}}{\sqrt{n} S_{(i-1)/n}^{(\alpha_n)}} \right)^2 + \frac{1}{3} \theta^3 \right) \right]^2 I_{A_m \cap B_n}, \\
&\quad \text{for } \theta \in \left(0, \frac{Y_{i,n}}{\sqrt{n} S_{(i-1)/n}^{(\alpha_n)}} \right).
\end{aligned} \tag{3.3}$$

By lemma 2, one can find

$$c_m \in \left(0, \frac{1}{m} \right] \text{ such that for large } n, S_{i/n}^{(\alpha_n)} \geq c_m \text{ for all } i = 0, 1, 2, \dots, n. \tag{3.4}$$

Define

$$\phi_{c_m}(x, y) = \begin{cases} \left(\frac{y}{x} \right)^2, & \text{when } x \geq c_m; \\ \left(\frac{3}{c_m^4} x^2 - \frac{8}{c_m^3} x + \frac{6}{c_m^2} \right) y^2, & \text{when } x < c_m. \end{cases} \tag{3.5}$$

Note in particular that ϕ_{c_m} is a function satisfying Hypothesis L_r in Delattre and Jacod (1997) with $r = 2$.

For n large enough, by Lemma 2 and Lemma 3, (3.3) can be rewritten as

$$\begin{aligned}
V^n I_{A_m \cap B_n} &\leq \frac{1}{n} \sum_{i=1}^n \phi_{c_m}(S_{(i-1)/n}^{(\alpha_n)}, Y_{i,n}) I_{A_m \cap B_n} + O \left(\frac{(\log n)^3}{n^{1/2}} \right) I_{A_m \cap B_n} \\
&= U(n, \phi_{c_m}) I_{A_m \cap B_n} + O \left(\frac{(\log n)^3}{n^{1/2}} \right) I_{A_m \cap B_n},
\end{aligned}$$

where $U(\cdot, \cdot)$ is defined in (3.2).

Furthermore,

$$\begin{aligned} V^n I_{A_m} &= V^n I_{A_m \cap B_n} + V^n I_{A_m \cap B_n^c} \\ &\leq U(n, \phi_{c_m}) I_{A_m} + (V^n - U(n, \phi_{c_m})) I_{A_m \cap B_n^c} + O\left(\frac{(\log n)^3}{n^{1/2}}\right) I_{A_m \cap B_n} \\ &= U(n, \phi_{c_m}) I_{A_m} + o_p(1) \quad (\text{by Lemma 1}). \end{aligned}$$

By Delattre and Jacod (1997),

$$U(n, \phi_{c_m}) \rightarrow_P \begin{cases} \int_0^1 \int_0^1 \int h(y) \phi_{c_m}(S_t, \beta[u + y\sigma_t S_t/\beta]) dy du dt, & \text{if } \beta > 0; \\ \int_0^1 \int h(y) \phi_{c_m}(S_t, y\sigma_t S_t) dy dt, & \text{if } \beta = 0. \end{cases}$$

Note that since $c_m \leq 1/m$, we have

$$\phi_{c_m}(S_{(i-1)/n}^{(\alpha_n)}, Y) = \left(\frac{Y}{S_{(i-1)/n}^{(\alpha_n)}} \right)^2 I_{A_m} + \phi_{c_m}(S_{(i-1)/n}^{(\alpha_n)}, Y) I_{A_m^c}.$$

Lemma 4 gives, when $\beta > 0$,

$$U(n, \phi_{c_m}) I_{A_m} \rightarrow_P \int_0^1 \frac{1}{S_t^2} \left(\sigma_t^2 S_t^2 + \frac{\beta^2}{6} - \frac{\beta^2}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{k^2} \exp\left\{-2\pi^2 k^2 \frac{\sigma_t^2 S_t^2}{\beta^2}\right\} \right) dt I_{A_m}.$$

It is easy to check that the above convergence is also true when $\beta = 0$.

Therefore, for $\beta \in [0, \infty)$,

$$\begin{aligned} &V^n I_{A_m} \\ &= U(n, \phi_{c_m}) I_{A_m} + o_p(1) \\ &\rightarrow_P \int_0^1 \frac{1}{S_t^2} \left(\sigma_t^2 S_t^2 + \frac{\beta^2}{6} - \frac{\beta^2}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{k^2} \exp\left\{-2\pi^2 k^2 \frac{\sigma_t^2 S_t^2}{\beta^2}\right\} \right) dt I_{A_m}. \end{aligned}$$

That is to say, for any $\delta > 0$, $\epsilon > 0$, there exists N , such that for all $n > N$,

$$P(|V^n I_{A_m} - \int_0^1 \frac{1}{S_t^2} (\sigma_t^2 S_t^2 + \frac{\beta^2}{6} - \frac{\beta^2}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{k^2} \exp\{-2\pi^2 k^2 \frac{\sigma_t^2 S_t^2}{\beta^2}\}) dt I_{A_m}| > \delta) < \epsilon.$$

On the other hand, since $A_m \nearrow \Omega$, there exists M large, such that

$$P(A_M^c) < \epsilon.$$

Therefore, for $n > N$,

$$\begin{aligned} & P(|V^n - \int_0^1 \frac{1}{S_t^2} (\sigma_t^2 S_t^2 + \frac{\beta^2}{6} - \frac{\beta^2}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{k^2} \exp\{-2\pi^2 k^2 \frac{\sigma_t^2 S_t^2}{\beta^2}\}) dt| > \delta) \\ & \leq P(A_M^c) + \\ & P\left(|V^n I_{A_M} - \int_0^1 \frac{1}{S_t^2} (\sigma_t^2 S_t^2 + \frac{\beta^2}{6} - \frac{\beta^2}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{k^2} \exp\{-2\pi^2 k^2 \frac{\sigma_t^2 S_t^2}{\beta^2}\}) dt I_{A_M}| > \delta\right) \\ & < 2\epsilon. \end{aligned}$$

This proves Theorem 1.

3.3 Proof of Theorem 2 and Theorem 3

By (3.3), for large n ,

$$\begin{aligned} & \sqrt{n} V^n I_{A_m \cap B_n} \\ & = \sqrt{n} \frac{1}{n} \sum_{i=1}^n \left[\sqrt{n} \left(\frac{Y_{i,n}}{\sqrt{n} S_{(i-1)/n}^{(\alpha_n)}} - \frac{1}{2} \left(\frac{Y_{i,n}}{\sqrt{n} S_{(i-1)/n}^{(\alpha_n)}} \right)^2 + \frac{1}{3} \theta^3 \right) \right]^2 I_{A_m \cap B_n}, \quad (3.6) \\ & \text{for } \theta \in \left(0, \frac{Y_{i,n}}{\sqrt{n} S_{(i-1)/n}^{(\alpha_n)}}\right). \end{aligned}$$

Using the $c_m \in (0, \frac{1}{m}]$ as in (3.4), we define

$$\psi_{c_m}(x, y) = \begin{cases} (\frac{y}{x})^3, & \text{when } x \geq c_m; \\ (\frac{4}{c_m^3} - \frac{3x}{c_m^4})y^3, & \text{when } x < c_m. \end{cases} \quad (3.7)$$

(3.6) can be further written as

$$\begin{aligned} & \sqrt{n}V^n I_{A_m \cap B_n} \\ & \leq \sqrt{n}U(n, \phi_{c_m})I_{A_m \cap B_n} - U(n, \psi_{c_m})I_{A_m \cap B_n} + O(\frac{(\log n)^4}{n^{1/2}})I_{A_m \cap B_n}; \end{aligned}$$

and

$$\begin{aligned} & \sqrt{n}V^n I_{A_m} \\ & = \sqrt{n}V^n I_{A_m \cap B_n} + \sqrt{n}V^n I_{A_m \cap B_n^c} \\ & \leq (\sqrt{n}U(n, \phi_{c_m}) - U(n, \psi_{c_m}))I_{A_m} \\ & \quad + (\sqrt{n}V^n - \sqrt{n}U(n, \phi_{c_m}) + U(n, \psi_{c_m}))I_{A_m \cap B_n^c} + O(\frac{(\log n)^4}{n^{1/2}})I_{A_m \cap B_n} \\ & = \sqrt{n}U(n, \phi_{c_m})I_{A_m} - U(n, \psi_{c_m})I_{A_m} + o_p(1), \end{aligned}$$

where ϕ_{c_m} is defined in (3.5), ψ_{c_m} in (3.7) and $U(\cdot, \cdot)$ in (3.2), and we have used Lemma 3 in the above.

Note that $\psi_{c_m}(S_t, \sigma_t S_t y)$ is an odd function of y , and $\beta = 0$; by Delattre and Jacod (1997),

$$U(n, \psi_{c_m}) \rightarrow_P \int_0^1 \int h(y) \psi_{c_m}(S_t, \sigma_t S_t y) dy dt = 0.$$

Therefore,

$$U(n, \psi_{c_m})I_{A_m} \rightarrow_P 0.$$

As a consequence,

$$\sqrt{n}V^n I_{A_m} = \sqrt{n}U(n, \phi_{c_m})I_{A_m} + o_p(1). \quad (3.8)$$

Also by Delattre and Jacod (1997), since $\phi_{c_m}(x, y)$ is even in y ,

$$\begin{aligned} & \sqrt{n}[U(n, \phi_{c_m}) - \int_0^1 \Gamma \phi_{c_m}(S_t, \beta_n) dt] \\ & \rightarrow \text{stably in law } \int_0^1 \Delta(\phi_{c_m}, \phi_{c_m})(S_t, 0)^{1/2} dB_s, \end{aligned} \quad (3.9)$$

where $B \perp W$, and

$$\begin{aligned} & \Gamma \phi_{c_m}(S_t, \beta_n) \\ &= \int_0^1 \int h(y) \phi_{c_m}(S_t, \beta_n [u + y \sigma_t S_t / \beta_n]) dy du \\ &= \int_0^1 \int h(y) \left(\left(\frac{\beta_n [u + y \sigma_t S_t / \beta_n]}{S_t} \right)^2 I_{A_m} + \phi_{c_m}(S_t, \beta_n [u + y \sigma_t S_t / \beta_n]) I_{A_m^c} \right) dy du \\ &= (\sigma_t^2 + \frac{\beta_n^2}{6} \frac{1}{S_t^2} - \frac{\beta_n^2}{\pi^2} \frac{1}{S_t^2} \sum_{k=1}^{\infty} \frac{1}{k^2} \exp\{-2\pi^2 k^2 \frac{\sigma_t^2 S_t^2}{\beta_n^2}\}) I_{A_m} + \\ & \int_0^1 \int h(y) \phi_{c_m}(S_t, \beta_n [u + y \sigma_t S_t / \beta_n]) dy du I_{A_m^c} \quad (\text{by Lemma 4}) ; \end{aligned} \quad (3.10)$$

and

$$\begin{aligned} & \Delta(\phi_{c_m}, \phi_{c_m})(S_t, 0) \\ &= \int h_{\sigma_t S_t}(y) \phi_{c_m}^2(S_t, y) dy - \left(\int h_{\sigma_t S_t}(y) \phi_{c_m}(S_t, y) dy \right)^2 \\ &= \int h_{\sigma_t S_t}(y) \left[\left(\frac{y}{S_t} \right)^4 I_{A_m} + \phi_{c_m}^2(S_t, y) I_{A_m^c} \right] dy \\ & \quad - \left(\int h_{\sigma_t S_t}(y) \left[\left(\frac{y}{S_t} \right)^2 I_{A_m} + \phi_{c_m}(S_t, y) I_{A_m^c} \right] dy \right)^2 \\ &= \left[\int h_{\sigma_t S_t}(y) \left(\frac{y}{S_t} \right)^4 dy - \left(\int h_{\sigma_t S_t}(y) \left(\frac{y}{S_t} \right)^2 dy \right)^2 \right] I_{A_m} + \\ & \quad \left[\int h_{\sigma_t S_t}(y) \phi_{c_m}(S_t, y)^2 dy - \left(\int h_{\sigma_t S_t}(y) \phi_{c_m}(S_t, y) dy \right)^2 \right] I_{A_m^c}; \end{aligned}$$

hence

$$\begin{aligned}
& \Delta(\phi_{c_m}, \phi_{c_m})(S_t, 0)^{1/2} \\
&= [\int h_{\sigma_t S_t}(y) (\frac{y}{S_t})^4 dy - (\int h_{\sigma_t S_t}(y) (\frac{y}{S_t})^2 dy)^2]^{1/2} I_{A_m} + \\
& \quad [\int h_{\sigma_t S_t}(y) \phi_{c_m}(S_t, y)^2 dy - (\int h_{\sigma_t S_t}(y) \phi_{c_m}(S_t, y) dy)^2]^{1/2} I_{A_m^c} \\
&= (2\sigma_t^4)^{1/2} I_{A_m} + [\int h_{\sigma_t S_t}(y) \phi_{c_m}(S_t, y)^2 dy - (\int h_{\sigma_t S_t}(y) \phi_{c_m}(S_t, y) dy)^2]^{1/2} I_{A_m^c}.
\end{aligned} \tag{3.11}$$

Plugging (3.10) and (3.11) into (3.9), and note that by the assumption that $\beta_n = O(n^{-\gamma})$,

$$\sqrt{n} \frac{\beta_n^2}{\pi^2} \int_0^1 \frac{1}{S_t^2} \sum_{k=1}^{\infty} \frac{1}{k^2} \exp\{-2\pi^2 k^2 \frac{\sigma_t^2 S_t^2}{\beta_n^2}\} dt \rightarrow 0 \text{ a.s. on } A_m \text{ as } n \rightarrow \infty.$$

One has,

$$\begin{aligned}
& \sqrt{n}[U(n, \phi_{c_m}) - (\int_0^1 \sigma_t^2 dt + \frac{\beta_n^2}{6} \int_0^1 \frac{1}{S_t^2} dt)] I_{A_m} \\
& + \sqrt{n}[U(n, \phi_{c_m}) - \int_0^1 \Gamma \phi_{c_m}(S_t, \beta_n) dt] I_{A_m^c} \\
& \rightarrow \text{stably in law} \\
& Z I_{A_m} + \int_0^1 [\int h_{\sigma_t S_t}(y) \phi_{c_m}(S_t, y)^2 dy - (\int h_{\sigma_t S_t}(y) \phi_{c_m}(S_t, y) dy)^2]^{1/2} dB_s I_{A_m^c},
\end{aligned}$$

where $Z \sim \int_0^1 (2\sigma_t^4)^{1/2} dB_s$, $B \perp W$.

For any continuous function g that vanishes outside a compact set, the above stable convergence implies that $\forall E \in \mathcal{F}$,

$$\begin{aligned}
& E[g(\sqrt{n}[U(n, \phi_{c_m}) - (\sigma_t^2 + \frac{\beta_n^2}{6} \int_0^1 \frac{1}{S_t^2} dt)] I_{A_m}) I_{A_m} I_E] \\
& \rightarrow E[g(\int_0^1 (2\sigma_t^4)^{1/2} dB_s I_{A_m}) I_{A_m} I_E].
\end{aligned} \tag{3.12}$$

And by defining $\eta_{c_m}(\cdot, \cdot)$ to be

$$\eta_{c_m}(x, y) = \begin{cases} (\frac{1}{x})^2, & \text{when } x \geq c_m; \\ (\frac{3}{c_m^4}x^2 - \frac{8}{c_m^3}x + \frac{6}{c_m^2}), & \text{when } x < c_m, \end{cases}$$

one has,

$$V_0^n I_{A_m} = V^n I_{A_m} - \frac{\beta_n^2}{6} U(n, \eta_{c_m}) I_{A_m}. \quad (3.13)$$

Again, by Delattre and Jacod (1997),

$$U(n, \eta_{c_m}) I_{A_m} = \frac{1}{n} \sum_{i=1}^n \frac{1}{(S_{i/n}^{(\alpha_n)})^2} I_{A_m} \rightarrow_P \int_0^1 \frac{1}{S_t^2} dt I_{A_m}.$$

and

$$\sqrt{n} \left(\frac{\beta_n^2}{6} U(n, \eta_{c_m}) - \frac{\beta_n^2}{6} \int_0^1 \frac{1}{S_t^2} dt \right) I_{A_m} = O_P(\beta_n^2) = o_P(1). \quad (3.14)$$

By (3.8), (3.13) and (3.14),

$$\sqrt{n} V_0^n I_{A_m} = \sqrt{n} \left(U(n, \phi_{c_m}) - \frac{\beta_n^2}{6} \int_0^1 \frac{1}{S_t^2} dt \right) I_{A_m} + o_p(1).$$

Also since that g is uniformly continuous, $\forall E \in \mathcal{F}$,

$$\begin{aligned} & \lim_{n \rightarrow \infty} E \left[g \left(\sqrt{n} \left(V_0^n - \int_0^1 \sigma_t^2 dt \right) I_{A_m} \right) I_{A_m} I_E \right] \\ &= \lim_{n \rightarrow \infty} E \left[g \left(\sqrt{n} \left[U(n, \phi_{c_m}) - \left(\int_0^1 \sigma_t^2 dt + \frac{\beta_n^2}{6} \int_0^1 \frac{1}{S_t^2} dt \right) \right] I_{A_m} \right) I_{A_m} I_E \right] \\ &= E \left[g \left(\int_0^1 (2\sigma_t^4)^{1/2} dB_t I_{A_m} \right) I_{A_m} I_E \right] \quad (\text{by (3.12)}), \end{aligned}$$

which implies, for any $\epsilon > 0$, there exists N , such that $\forall n \geq N$,

$$|E[g(\sqrt{n}[V_0^n - \sigma_t^2]I_{A_M})I_{A_M}I_E] - E[g(\int_0^1 (2\sigma_t^4)^{1/2}dB_t I_{A_M})I_{A_M}I_E]| < \epsilon.$$

Note also that g is bounded, suppose $|g| \leq M_g$. Recall that $P(A_M^c) \rightarrow 0$, one can choose M such that $P(A_M^c) < \epsilon/M_g$.

So for $n \geq N$,

$$\begin{aligned} & |E[g(\sqrt{n}[V_0^n - \int_0^1 \sigma_t^2 dt])I_E] - E[g(\int_0^1 (2\sigma_t^4)^{1/2}dB_t)I_E]| \\ \leq & |E[g(\sqrt{n}[V_0^n - \int_0^1 \sigma_t^2 dt]I_{A_M})I_{A_M}I_E] - E[g(\int_0^1 (2\sigma_t^4)^{1/2}dB_t I_{A_M})I_{A_M}I_E]| + 2Mg * P(A_M^c) \\ \leq & 3\epsilon \end{aligned}$$

Hence we've proved that for all continuous function g that vanishes outside a compact set, $\forall E \in \mathcal{F}$,

$$\lim_{n \rightarrow \infty} E[g(\sqrt{n}[V_0^n - \int_0^1 \sigma_t^2 dt])I_E] = E[g(\int_0^1 (2\sigma_t^4)^{1/2}dB_t)I_E],$$

i.e.,

$$\sqrt{n}[V_0^n - \int_0^1 \sigma_t^2 dt] \rightarrow_{\mathcal{L}\text{-stably}} \int_0^1 (2\sigma_t^4)^{1/2}dB_t.$$

This finishes the proof of Theorem 3. The proof of Theorem 2 is basically contained in the proof above.

3.4 The Case of General μ_t and σ_t

Step 1: For general cases when $\mu_t \neq 0$, if there exists $L_\sigma, U_\sigma, C_\mu \in (0, \infty)$, such that $L_\sigma \leq \sigma_t \leq U_\sigma$ and $|\mu_t| \leq C_\mu$ for $t \in [0, 1]$, the previous results all hold.

For the simplicity of notation, we consider the log scale. Let P be the probability

measure corresponding to the system

$$dX_t = \sigma_t dW_t$$

and Q the probability measure corresponding to the system

$$dX_t = \mu_t dt + \sigma_t dW_t^Q,$$

where W_t and W_t^Q are standard Brownian Motions under P and Q respectively.

Note that by the Girsanov Theorem (see, for example, page 164 of Øksendal (2003)), for bounded σ_t and μ_t (as stated in the conditions of “Step 1”), P and Q are mutually absolutely continuous.

The following proposition justifies the conclusion of “Step 1”.

Proposition (Mykland and Zhang (2009)) *Suppose that ζ_n is a sequence of random variables which converges stably to $N(b, a^2)$ under P (meaning that $N(b, a^2) = b + aN(0, 1)$, where $N(0, 1)$ is a standard normal variable independent of \mathcal{F} , also a and b are \mathcal{F} measurable). Then ζ_n converges stably in law to $b + aN(0, 1)$ under Q , where $N(0, 1)$ remains independent of \mathcal{F} under Q .*

Step 2: for locally bounded σ_t and μ_t , the stable convergence and the convergence in probability stay valid.

This can be proved by a localization argument which uses essentially the same techniques as in the derivation in the last part of section 3.3. For example, to unbound σ_t , one considers a sequence of stopping times τ_m corresponding to a sequence of positive constants σ_m which increases to infinity as $m \rightarrow \infty$: $\tau_m = \min\{t : \sigma_t^2 \geq \sigma_m^2\}$, and note the fact that the sets $\{\tau_m > T\} \nearrow \Omega$.

In particular, the locally bounded assumption is automatically satisfied when σ_t and μ_t are continuous.

3.5 Proof of Theorem 4

Similar argument as the Proof of Theorem 3 in Li and Mykland (2007) gives the result.

4 CONCLUSIONS AND DISCUSSION

In summary, we have proved the following results:

Under the condition that $\sqrt{n}\alpha_n \rightarrow \beta \in [0, \infty)$,

$$V^n \rightarrow_P \int_0^1 \sigma_t^2 dt + \frac{\beta^2}{6} \int_0^1 \frac{1}{S_t^2} dt - \frac{\beta^2}{\pi^2} \int_0^1 \frac{1}{S_t^2} \sum_{k=1}^{\infty} \frac{1}{k^2} \exp \left\{ -2\pi^2 k^2 \frac{\sigma_t^2 S_t^2}{\beta^2} \right\} dt.$$

And under the condition that $\beta_n = O(n^{-\gamma})$ for certain $\gamma > 0$, we have

$$\sqrt{n} \left(V^n - \int_0^1 \sigma_t^2 dt - \frac{\beta_n^2}{6} \int_0^1 \frac{1}{S_t^2} dt \right) \rightarrow_{\mathcal{L}\text{-stably}} \int_0^1 \sqrt{2} \sigma_t^2 dB_t,$$

and

$$\sqrt{n} \left(V^n - \frac{\alpha_n^2}{6} \sum_{i=1}^n \frac{1}{(S_{i/n}^{(\alpha_n)})^2} - \int_0^1 \sigma_t^2 dt \right) \rightarrow_{\mathcal{L}\text{-stably}} \int_0^1 \sqrt{2} \sigma_t^2 dB_t,$$

where B is a Brownian Motion independent with W .

We have used the later result to create a bias-correction that works for “small rounding” by defining the bias-corrected estimator to be

$$V_0^n = V^n - \frac{\alpha_n^2}{6} \sum_{i=1}^n \frac{1}{(S_{i/n}^{(\alpha_n)})^2}.$$

When $\alpha_n = \alpha$, $\beta_n = \alpha_n \sqrt{n} \rightarrow \infty$, V^n blows up to infinity at a rate being square root of the sample size n :

$$\frac{1}{\sqrt{n}} V^n \rightarrow_P \frac{1}{\sigma} \sqrt{\frac{2}{\pi}} \sum_{k=1}^{\infty} L_1^{\log((k+1)\alpha)} \left(\log \frac{k+1}{k} \right)^2,$$

where L_t^a is the local time of the continuous semimartingale $X_t = \log S_t$.

Note that while we work with observations on a time interval $[0, 1]$, results for the more general case of time interval $[0, T]$ is obtained by rescaling. The case of unequal observation times can be studied using the methods of Jacod and Protter (1998) and Mykland and Zhang (2006).

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