# Industrial Dynamics, Endowment Structure, and Economic Growth

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**Abstract**

Motivated by the empirical patterns of industrial dynamics at the disaggregated level, we propose a theory of endowment-driven structural transformation by developing a tractable growth model with infinite industries. We show that, while the aggregate behaviors are still consistent with the Kaldor facts, the underlying industries exhibit inverse-V-shaped life cycles. As the capital accumulates and reaches a certain threshold, a new industry appears, prospers, and then declines. While the industry is declining, a more capital-intensive industry appears, booms, and eventually declines, *ad infinitum*. Explicit solutions are obtained to fully characterize the whole process of perpetual structural change and economic growth.

**Key Words:** Industrial Dynamics, Economic Growth, Structural Transformation, Factor Endowment

**JEL Codes:** O14, O40

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"The growth of GDP may be measured up in the macroeconomic treetops, but all the action is in the microeconomic undergrowth, where new limbs sprout, and dead wood is cleared away."

–*World Bank Commission on Growth and Development* (2008, pp.2-3)

# 1 Introduction

The recent research on structural transformation has mainly focused on the composition shift across very aggregate sectors. For example, three-sector growth models are constructed to reconcile the Kaldor facts[1] with the Kuznets facts, the latter of which refer to the pattern that the agriculture share in GDP has a secular decline, the industry share demonstrates a hump shape, and the service sector share increases.[2] Some other research uses two-sector models to study the process of industrialization[3] or the expansion of the service sector in developed economies.[4]

However, the structural transformation at the disaggregated industry level, which we call *industrial dynamics* in this paper, is much less studied both empirically and theoretically in the recent literature. To have a more concrete idea, let us first look at four examples of HP-filtered industrial dynamics shown in Figure 1.

[Insert Figure 1 here]

The vertical axis represents the employment share of an industry (the ratio of industrial employment to total manufacturing employment) and the horizontal axis

[1] Kaldor facts refer to the relative constancy of the growth rate of total output, the capital-output ratio, the real interest rate, and the share of labor income in GDP.

[2] See, for example, Kongsamut, Rebelo and Xie (2001), Ngai and Pissarides (2007), Acemoglu and Guerrieri (2008), Herrendorf, Rogerson and Valentinyi (2009) for closed-economy models and empirical studies. Matsuyama (2009) and Yi and Zhang (2010) study the structural transformation in an open economy. Earlier discussion on the Kuznets facts includes Chenery (1960), Kuznets (1966), Maddison (1980), *etc.*.

[3] See, for example, Hansen and Prescott (2002); Lucas (2004, 2009), Wang and Xie (2004), Duarte and Restuccia (2009).

[4] See, for example, Lee and Wolpin (2006), Buera and Kaboski (2009, 2011), *etc.*

is time. This figure uses the NBER-CES Manufacturing Industry Data for the US, which adopts the six-digit NAICS codes and covers 473 industries within the manufacturing sector from 1958 to 2005. Figure 1A depicts the dynamics of Industry 326199 (All Other Plastics Product Manufacturing), where the employment share increased steadily from 0.5% of total manufacturing employment in 1958 to 3.5% in 2005. The dynamics of Industry 313210 (Broad Woven Fabric Mills) is depicted in Figure 1B, where its employment share decreased steadily from 2.5% to 0.5% over the same period. Figure 1C plots the dynamics of Industry 334111 (Electronic Computer Manufacturing), whose employment share demonstrates an inverse V-shaped pattern, increasing from 1958 to 1985 and then declining afterwards. Lastly, Figure 1D shows that the employment share of Industry 339943 (Marking Device Manufacturing) has multiple peaks, reaching local maximums in 1972 and 1996.

Are there any regular patterns of industrial dynamics at such a disaggregated level? Which industries should we expect to expand (or decline) in a country and why? How long would a leading industry maintain the predominant role? What fundamental forces drive the industrial dynamics? What is the relationship between the industrial dynamics and macroeconomic performance? To address important questions like these, we must go beyond the current two-sector or three-sector framework. The purpose of this paper, therefore, aims to document the empirical patterns of industrial dynamics at a disaggregated level of industries, and to develop a theoretical framework to help explain the observed patterns of industrial dynamics together with the aggregate dynamics.

We document four important empirical facts for the industrial dynamics at disaggregated levels. First, there exists tremendous cross-industry heterogeneity both in capital intensities and total factor productivities (TFPs). Second, the employment share (or the value-added share) of an industry typically exhibits an inverse-V-shaped life cycle pattern, namely, an industry first expands, reaches the peak, and eventually declines. Third, the further away an industry's factor intensity deviates from the factor endowment of the economy, the smaller is the employment share (or output share) of this industry in the total GDP, which we may call the *congruence fact*.[5] Fourth, the more capital-intensive an industry, the later it reaches its peak. Similar patterns are also found in the UNIDO data set, which covers 148 countries and 18 manufacturing sectors from 1963 to 2006 (Haraguchi and Rezonja (2010)). In fact, the documentation and analysis of such patterns of industrial dynamics can be dated at least back to the early 1960s. For example, Chenery (1960), Chenery and Taylor (1968) and Chenery, Robinson, and Syrquin (1986) show that the major products in the manufacturing sector gradually shift from the labor-intensive ones to more capital-intensive ones as the economy develops. The waxing-and-waning pattern of industrial development was sometimes also referred to as the flying geese pattern of economic development by Akamatsu (1962) and further verbally expatiated by Ozawa (2009).

All these empirical findings and existing informal analyses suggest that the endowment structure (defined as the relative factor abundance) plays an important role in both determining the industrial structure and driving the industrial dynamics.[6] Therefore, we are motivated to propose a formal theory of endowment-driven structural transformation at the disaggregated levels. To understand the basic intuition, we may appeal to the Rybczynski theorem in the trade theory,[7] which states that in a static model with two goods and two factors, the capital-intensive sector expands while the labor-intensive sector shrinks as the economy becomes more capital abundant. However, to simultaneously explain all four empirical features,

[5] Empirically, this coherence fact also applies for the growth rate, please refer to Lin (2003), who examines the cross-country data that incorporates both the developing and developed countries. Che (2010) utilizes the industrial data for the OECD countries.

[6] In his Marshall Lectures, Lin (2009) proposes that many development issues including growth, inequality, industrial policies and so on can all be better understood by analyzing the congruence of the industrial structure with the comparative advantages determined by the endowment structure and its change. There, the endowment structure is more formally defined as the composition of the production factors (including labor, human capital, physical capital, land and other natural resources), all of which can be potentially allocated through market mechanism, as well as the soft and hard infrastructures, which are usually provided by the government (Lin, 2011).

[7] Rybczynski theorem holds in both closed and open economies.

we have to face at least two major challenges. First, it requires a model with many (ideally an infinite number of) industries to analyze the industrial dynamics at sufficiently disaggregate levels, but the Rybczynski Theorem says very little about how the production structures change when the number of goods exceeds two (see, Feenstra, 2004). Second, and more importantly, industrial dynamics should be studied in a dynamic general equilibrium model with endogenous capital accumulation, while the aggregate growth in the model needs to be consistent with the Kaldor facts (such as the constant aggregate growth rate and relatively stable capital share). Yet it is well recognized as technically difficult to characterize analytically the reallocation process across different sectors (industries) and the transitional dynamics even in a growth model with only two sectors. Now we need to characterize the possibly nonlinear "life cycle" dynamics of all the infinite industries simultaneously and also relate them to the dynamics of the aggregate economy in an infinite horizon, so the high-dimension problem may appear even more unwieldy. Fortunately, despite these challenges, our model enables us to obtain closed-form solutions to characterize the whole dynamic process of the perpetual inverse-V-shaped industrial dynamics for each industry, while the aggregate economy still matches the Kaldor facts.

We first develop a static model with infinite industries (or goods, interchangeably) and two factors (labor and capital). With a general CES production function for the final commodity, we obtain a version of the *generalized Rybczynski theorem*: For any given endowment of capital and labor, there exists a cutoff industry such that all the industries that are more capital intensive than this cutoff industry will increase their output as the capital endowment marginally increases, while all the industries less capital intensive than this cutoff industry will shrink. Moreover, this cutoff industry itself shifts toward the more capital-intensive direction as the capital endowment increases. As a special case, when the CES substitution elasticity is infinity, we show analytically that there generically exist only two industries in the competitive equilibrium, where the capital intensities of the active industries are the closest to the capital-labor ratio of the economy. That result is consistent with the *congruence fact*. The model implies that the underlying industries are endogenously different at different levels of economic development (measured by different capital-labor ratios in the endowment of the economy).

Then the model is extended to a dynamic environment where capital accumulates endogenously over time. The dynamic decision can be decomposed into two steps. First, a representative household optimizes the intertemporal allocation of capital expenditure, which determines the evolution of capital endowment in the economy. Then, at each time point the production structures are determined by the capital and labor endowments the same way as in the static model. Endogenous changes in the industrial structures translate into different functional forms of the capital accumulation function; therefore, ultimately, the mathematical problem is to solve a Hamiltonian system with endogenously switching state equations because of the endogenous structural transformation. When the CES substitution elasticity is infinity (linear case), we show that there always exists a unique aggregate growth path with a constant growth rate, along which the underlying industries shift over time by following an inverse-V-shaped pattern. We also characterize the speed of industrial upgrading and the whole span of the "life cycle" of each industry. As capital accumulates and thus becomes relatively cheaper over time, the more capital-intensive goods are produced, gradually replacing the more labor-intensive goods. This process continues forever, which generates the endless inverse-V-shaped industrial dynamics.

Our paper is most closely related to two papers in the literature. Acemoglu and Guerrieri (2008) develop a two-sector growth model to examine how capital deepening has an asymmetric impact on the sectors with different capital intensities, but their model does not specifically explain the repetitive inverse-V-shaped industrial upgrading. Moreover, their focus is not on the industrial dynamics *per se*, but rather on the long-run *asymptotic* growth rates. By contrast, we highlight the life-cycle industrial dynamics at the disaggregated levels *during* the whole process of structural transformation. Ngai and Pissarides (2007) emphasize how the exogenously unbalanced productivity growth across different sectors drives the structural transformation, and they assume that different sectors have the same capital intensities in the production function. Our model, as motivated by the empirical facts, assumes that different industries are different in their capital intensities. We highlight the improvement of endowment structure as the fundamental mechanism that drives the structural transformation, which complements the literature of productivity-driven structural change.[8]

Another strand of literature emphasizes demand-driven structural transformation, which typically assumes non-homothetic preferences (such as the Stone-Geary or hierarchic utility functions).[9] The implication is that demand shifts across the consumption goods as people get richer and hence structural transformation occurs. None of these papers explicitly attempts to characterize the aforementioned inverse-V-shaped industrial dynamics and the mechanism is also different from ours.[10] Our model assumes for simplicity that all the technologies are freely available, different from the R&D-driven structural transformation (creative destruction), which emphasizes how new products are invented.[11] That mechanism seems less relevant for developing countries than the developed ones at the technology frontier. Stokey (1988) characterizes how learning by doing keeps the band of the produced commodities moving toward

[8] The productivity-driven structural transformation is also discussed in the Baumol (1967), Hansen and Prescott (2002), *etc.* To understand the endogenous productivity increases across different sectors, Acemoglu (2007) argues that technology progress is often biased toward utilizing the more abundant production factors, which is consistent with the empirical evidences we find with the US data. However, we show that the industrial dynamics can be solely driven by capital accumulation in our model even if the productivities are identical across industries.

[9] See, for example, Laitner (2000), Caselli and Coleman (2001), Kongsamut, Rebelo, and Xie (2001), Gollin, Parente, and Rogerson (2007), and Foellmi and Zweimuller (2008).

[10] Herrendorf, Rogerson and Valentinyi (2009) explore how to empirically test and distinguish the productivity-driven structural transformation and the demand-driven structural transformation. In our model, different TFP levels across industries is mathematically isomorphic to the alternative interpretation that different industrial goods have different qualities and hence different income demand elasticities, but it is not the key force that drives the structural change.

[11] See Aghion and Howitt (1992), Grossman and Helpman (1991), *etc.*

higher and higher qualities. This literature, like other growth models, mainly focuses on the endogenous technical change that sustains the long-run balanced growth path while leaving the transitional dynamics and industrial dynamics largely aside.

To highlight the direct impact of endowment change on the industrial dynamics, we shut down the effect of international specialization by only considering a closed economy in this paper. We conjecture that our main proposition, namely, the change in endowment structures drives the industrial dynamics, will be presumably strengthened in an open economy as predicted by the Heckscher-Ohlin trade models.[12]

The paper is organized as follows: Section 2 documents and summarizes the empirical facts about industrial dynamics. Section 3 presents the static model. The dynamic model is analyzed in Sections 4. Section 5 considers the more general CES production function. Section 6 concludes. Technical proofs are in the Appendix.

# 2 Empirical Patterns of Industrial Dynamics

## 2.1 Evidence from US Data

To document the detailed features of the industrial dynamics, ideally we need a sufficiently long time series of production data for each industry at a sufficiently high-digit industry level. The best data set we have access to is the NBER-CES Manufacturing Industry Data set for the US, which adopts the 6-digit NAICS codes and covers 473 industries within the manufacturing sector from 1958 to 2005.[13]

First of all, we find that there exist tremendous variations both in the capital-labor ratios and TFPs across different industries.[14] For example, in 1958, the capital-labor

[12]For more discussions in dynamic Heckscher-Ohlin models, readers are guided to Chen (1992), Baxter (1992), Ventura (1997), Nishimura and Shimomura (2002), Bond, Trask and Wang (2003), Bajona and Kehoe (2006), and Caliendo (2010). There is also a vast literature on technology diffusion and location movement of industries across countries, which can be dated back at least to Akamatsu (1962), Vernon (1966) and Krugman (1979). Recent analysis includes Lucas (2009), etc.

[13]The data set can be downloaded from the website: http://www.nber.org/nberces

[14]Capital refers to "real capital stock" in the NBER-CES data set. The sectoral TFP in this data set is obtained from a five-factor production function: capital, production worker hours, non-production workers, non-energy materials, and energy. The sectoral TFP growth is calculated as the growth rate of output minus the revenue-share-weighted average of the growth rates of each of

ratio in the most capital-intensive industry is 522,510 US dollars per worker, which is 986 times larger than that in the least capital-intensive industry. In 2005 the capital-labor ratio in the most capital-intensive industry is still about 148 times higher than that in the most labor-intensive industry. When the TFP in 1997 is set to unity, the highest industry-specific TFP is 2.78 in 1958, which is 278 times larger than that in the least-efficient industry. In 2005 the top-bottom cross-industry TFP difference is still 15-fold large. Please refer to Table A1 in the Appendix for more detailed descriptive statistics.

To understand what determines the industrial structures and their dynamics, we first run a simple regression of an industry's employment share on both its TFP and a variable which measures the congruence between the industry's capital intensity and the aggregate endowment structure after controlling for the industrial fixed effect. We obtain the following results (with $t$ statistics in parentheses):

$$LS_{it} = 0.0016 - \underset{(4.7222)}{0.000085} \left| \frac{K_{it}/L_{it} - K_t/L_t}{K_t/L_t} \right| + \underset{(11.183)}{0.00052} TFP_{it}, \qquad (1)$$

where $LS_{it}$ is the ratio of industry $i$'s employment to the total manufacturing employment at year $t$, and $\left| \frac{K_{it}/L_{it} - K_t/L_t}{K_t/L_t} \right|$ is the absolute value of a normalized difference between industry $i$'s capital-labor ratio and the aggregate capital-labor ratio at year $t$. The result shows that an industry's employment share is significantly and positively correlated with the level of TFP. On the other hand, the negative coefficient before the term $\left| \frac{K_{it}/L_{it} - K_t/L_t}{K_t/L_t} \right|$ measures the magnitude of the congruence effect. It says that an industry's employment share becomes significantly smaller if the difference between the industry's capital intensity and the aggregate capital-labor ratio is larger, suggesting that if an industry's capital intensity (measured by its capital-labor ratio) and the economy's endowment structure (measured by the aggregate capital-labor ratio) are more congruent, the industry is relatively larger. Unfortunately,

the five inputs. The sector-specific shares are calculated as the industry-specific input expenditure divided by the industry's output, with the exponent on capital chosen to ensure that the production function is constant returns to scale.

when we use the capital expenditure, instead of capital-labor ratio, to measure the capital intensity in the above regression, the sign of the congruence effect becomes positive but not significant. The congruence effect seems not very robust at first glance.

A problem with regression (1) is that the industries in this data set are still not disaggregated enough for our purpose. Table A1 shows that the capital-labor ratio is increasing over time for almost every industry, which suggests that each of the 473 industries in the data set may consist of many different products which are produced with techniques that have different capital intensities. Hence the increase in the capital-labor ratio within an industry may reflect the fact that the production activity in this industry switches from the labor-intensive products/procedures/tasks to capital-intensive products/procedures/tasks. To tackle this problem, we follow Schott (2003) by re-categorizing the industries according to their capital-labor ratios. We first rank the 22,704 observations (473 industries by 48 years) according to their capital-labor ratios, and then divide all the observations into 99 categories with 225 observations each. We call the most labor-intensive group "industry 1" and the most capital-intensive group "industry 99". Figure 2 plots the time series of the HP-filtered employment shares of these 99 newly-defined industries from 1958 to 2005.

[Insert Figure 2 here]

Except for a few anomalies that seem affected by idiosyncratic shocks, a discernible pattern emerges from these graphs. That is, within this time window, the employment share of the most labor-intensive industries is declining over time, the employment share of the "middle" capital-intensive industries demonstrates a hump shape, and the employment share of the most capital-intensive ones is increasing over time. Moreover, the more capital-intensive the industry, the later it reaches the peak. If longer time series data become available, we would expect to see the rising stage of the most labor-intensive industries before 1958 and perhaps the eventual declines of the capital-intensive industries after 2005. We obtain similar patterns when the employment share is replaced with the value-added share and/or when the industrial capital intensity is measured alternatively by the share of capital expenditure (one minus the ratio of labor expenditure to the value added). We refer to this pattern as the *inverse-V-shaped pattern* of industrial dynamics.

For the newly defined industries, we now re-examine the correlations of "factor congruence" and industrial TFP with the industry's employment share and/or the value-added share. The results are reported in Table 1. After trying different alternative measures for the industry shares and capital intensities, we find that as an industry's capital intensity deviates more from the economy's endowment structure, the industry becomes significantly smaller. In addition, the industrial TFP level is significantly and positively correlated to the industry's employment share, but it has no significant correlation with the industry's value-added share.

[Insert Tables 1 and 2 here]

Next, we regress the peak time of an industry's share (either employment share or value added share) on its capital intensity (the time-average of the real capital stock to labor ratio or the share of the capital expenditure). The results are reported in Table 2. The regression results indicate that a more capital-intensive industry reaches its peak later.

Finally, we examine the correlation between the industrial capital-labor ratios and the TFPs. The descriptive statistics in Table A1 show that all the industries generally become more capital intensive and more productive as time goes by. We pool together the TFPs and capital-labor ratios across all the industries in all the years and run a simple regression of TFPs on the capital-labor ratios, which yields the following results (with $t$ statistics reported in parentheses):

$$TFP_{it} = 0.9217 + \underset{(4.000)}{4.63} \times 10^{-5}(K_{it}/L_{it}), \tag{2}$$

where $TFP_{it}$ is industry $i$'s TFP at year $t$ and $K_{it}/L_{it}$ is the capital-labor ratio in industry $i$ at year $t$. The result shows that industrial TFPs and capital intensities are positively correlated at the 1% significance level.

## 2.2 Evidence From Cross-Country Data

This subsection cites the empirical findings in Haraguchi and Rezonja (2010), who explore the UNIDO data set covering 135 countries from 1963 to 2006. Their data set adopts the 2-digit ISIC codes and entails 18 manufacturing sectors. Since different countries have data for different periods, and developing countries typically have much shorter time-series data, it is impossible to conduct time-series analysis country by country, so they group all the time periods together for their regressions. One of their most robust findings is that the log of the value-added share of an industry in the total GDP can be best fit by including a quadratic term of the log real GDP per capita in the regressions and the coefficient is both negative and significant after controlling various country-specific factors (such as population and natural resource abundance) and country dummies. This is consistent with the patterns of the inverse V-shaped industrial dynamics we documented earlier for the US data.

Figure 3 plots how the value-added share of an industry in the total GDP changes with the real GDP per capita (using 2005 US dollars) for the 18 industries respectively. It is copied from Haraguchi and Rezonja (2010).

[Insert Figure 3 here]

These industries are grouped in three separate panels. Panels a, b, and c are also referred to as the "early sectors", "middle sectors" and "late sectors", respectively, according to Chenery and Taylor (1968) based on the approximate sequence of development. Panel a is the most labor intensive group and Panel c is the most capital intensive group. The blue dots represent "small countries", defined as those with population less than 15 million in year 1983. The red dots represent "large countries" (with population over 15 million in 1983). It is clear that almost all the industries (except for the precision instruments) demonstrate an inverse-V-shaped pattern. Moreover, industries in Panel a reach the peak at a lower level of GDP per capita ("earlier") than those in Panel b, which in turn reach the peak "earlier" than those in Panel c. In other words, a more capital-intensive industry reaches the peak at a higher level of GDP per capita ("later").

### 2.3 Summary of Empirical Facts

Combining all the afore-mentioned empirical results together with other related research in the literature, we summarize the four important facts for the industrial dynamics in the manufacturing sector.

**Fact 1 (cross-industry heterogeneity):** There exists tremendous cross-industry heterogeneity in capital-labor intensities.

**Fact 2 (inverse-V-shaped dynamics):** An industry typically exhibits an inverse-V-shaped dynamic pattern: its employment share (or value-added share) first increases, reaches the peak, and then declines.

**Fact 3 (congruence fact):** The further away an industry's capital intensity deviates from the economy's endowment structure (measured either by the capital-labor ratio or capital expenditure), the smaller is the industry (both in terms of the employment share and the value-added share).

**Fact 4 (timing fact):** The more capital intensive an industry, the later it reaches its peak (in terms of the employment share or value-added share).

Next by assuming that different industries have different capital intensities, we will present a theoretical model that aims to simultaneously explain Facts 2, 3 and 4 at the industrial level while at the same time remaining compatible with the Kaldor facts at the aggregate level.

# 3 Static Model

## 3.1 Setup

Consider a closed economy with a unit mass of identical households and infinite industries. Each household is endowed with $L$ units of labor and $E$ units of physical capital, which can be easily extended to incorporate intangible capital as well. But for exposition convenience, let us simply call it physical capital. The representative household consumes a composite final commodity $C$, which is produced by combining all the intermediate goods $c_n$, where $n \in \{0, 1, 2, ...\}$. Each intermediate good should be interpreted as an industry, although we will use "good" and "industry" interchangeably throughout the paper.

For analytical simplicity, assume the production function of the final commodity is

$$C = \sum_{n=0}^{\infty} \lambda_n c_n, \tag{3}$$

where $\lambda_n$ represents the marginal productivity of good $n$ in the final good production.[15] We require $c_n \geq 0$ for any $n$. The representative household's utility function is $CRRA$:

$$U = \frac{C^{1-\sigma} - 1}{1-\sigma}, \text{ where } \sigma \in (0, 1]. \tag{4}$$

All the production technologies exhibit constant returns to scale. In particular, good 0 is produced with labor only. One unit of labor produces one unit of good 0, which also serves as the numeraire. To produce any good $n \geq 1$, both labor and

[15] It is not unusual in growth literature to assume perfect substitutability for the output across different production activities. For example, the agricutural Malthus production and the modern Solow production are two linearly additive components for the total output in Hansen and Prescott (2002) and Lucas (2009). The setup might also be interpreted as that $c_n$ denotes the consumption of good $n$ and $\lambda_n$ represents the quality of good $n$. We will relax the perfect substitutability assumption later.

capital are required and the production functions are Leontief[16]:

$$F_n(k, l) = \min\{\frac{k}{a_n}, l\}, \tag{5}$$

where $a_n$ measures the capital intensity of good $n$. All the markets are perfectly competitive. Let $p_n$ denote the price of good $n$. Let $r$ denote the rental price of capital and $w$ denote the wage rate. The zero profit condition for a firm implies that $p_0 = w$ and $p_n = w + a_n r$ for $n \geq 1$.

Since the data suggest that a more capital-intensive technology is generally more productive (refer to regression result (2)), we assume that both $a_n$ and $\lambda_n$ are increasing in $n$. To obtain closed-form solutions, let us assume

$$\lambda_n = \lambda^n,\ a_n = a^n. \tag{6}$$

$$\lambda > 1 \text{ and } a - 1 > \lambda. \tag{7}$$

$a > \lambda$ is imposed to rule out the trivial case that only the most capital-intensive good is produced in the static equilibrium, and we strengthen the assumption further to $a - 1 > \lambda$ to simplify the analysis as good 0 requires no capital.

The household problem is to maximize (4) subject to the following budget constraint

$$\sum_{n=0}^{\infty} p_n c_n = wL + rE. \tag{8}$$

### 3.2 Market Equilibrium

In the Appendix, we first establish that at most two goods are simultaneously produced in the equilibrium and that these two goods have to be adjacent in the capital intensities. The intuition is the following. Suppose goods $n$ and $n+1$ are produced for some $n \geq 1$. From the producer's point of view, to produce the final

[16] It can be shown that the key qualitative features will remain valid when the production function is changed to Cobb-Douglass, but it enhances the nonlinearity of the problem enormously, which makes it much harder to obtain closed-form solutions, especially for the dynamic analysis.

goods, the marginal rate of transformation (MRT) between the two intermediate goods must be equal to their price ratio:

$$MRT_{n+1,n} = \lambda = \frac{p_{n+1}}{p_n} = \frac{w + a^{n+1}r}{w + a^n r},$$

which yields

$$\frac{r}{w} = \frac{\lambda - 1}{a^n(a - \lambda)}. \tag{9}$$

Obviously, $MRT_{j,j+1} > \frac{p_j}{p_{j+1}}$ whenever $\frac{r}{w} > \frac{\lambda-1}{a^j(a-\lambda)}$, and $MRT_{j,j-1} > \frac{p_j}{p_{j-1}}$ whenever $\frac{r}{w} < \frac{\lambda-1}{a^{j-1}(a-\lambda)}$ for any $j = 1, 2, ...$ Therefore, when (9) holds, good $n+1$ must be strictly preferred to good $n+2$, because the MRT is larger than their price ratio. This means that $c_j = 0$ for all $j \geq n+2$. Using the same logic, we can also verify that $c_j = 0$ for all $1 \leq j \leq n-1$. In addition, condition (7) ensures that good 0 will not be produced. Similarly, when goods 0 and 1 are produced in an equilibrium, good 1 is strictly preferred to any good $n \geq 2$.

The market clearing conditions for labor and capital are

$$c_n + c_{n+1} = L, \tag{10}$$

$$c_n a^n + c_{n+1} a^{n+1} = E. \tag{11}$$

The market equilibrium can be illustrated graphically in Figure 4, where the horizontal and vertical axes are labor and capital, respectively. Point $O$ is the origin and Point $W = (L, E)$ denotes the endowment of the economy. When $a^n L < E < a^{n+1} L$, as shown in the current case, only goods $n$ and $n+1$ are produced. The factor market clearing conditions, (10) and (11), determine the equilibrium allocation of labor and capital in industries $n$ and $n+1$, which are represented respectively by vector $OA$ and vector $OB$ in the parallelogram $OAWB$. Lines $Oa^n = (1, a^n)\, c_n$ and $Oa^{n+1} = \left(1, a^{n+1}\right) c_{n+1}$ are the vectors of factors used in producing $c_n$ and $c_{n+1}$ in the equilibrium. If the capital increases so the endowment point moves from $W$ to

$W'$, the new equilibrium becomes parallelogram $OA'W'B'$ so that $c_n$ decreases but $c_{n+1}$ increases. When $E = a^n L$, only good $n$ is produced. Similarly, if $E = a^{n+1}L$, only good $n+1$ is produced.[17]

[Insert Figure 4 Here]

How $c_n$ changes with the endowment structure has already been encoded in Figure 4, but the implied inverse-V-shaped pattern can be more intuitively seen in Figure 5. When capital endowment $E$ reaches threshold value $a^n L$, good $n+1$ enters the market, and its output increases as $E$ increases up to the point $E = a^{n+1}L$ and then it declines. At the point $E = a^{n+2}L$, good $n+1$ exits from the market while a new good with a higher productivity $(n+3)$ appears.

[Insert Figure 5 Here]

More precisely, the equilibrium output of each good $c_n$, the relative factor prices $\frac{r}{w}$, and the corresponding aggregate output $C$ are summarized in the following table.

Table 3: Static Equilibrium

| $0 \leq E < aL$ | $a^n L \leq E < a^{n+1}L$ for $n \geq 1$ |
|---|---|
| $c_0 = L - \frac{E}{a}$ | $c_n = \frac{La^{n+1} - E}{a^{n+1} - a^n}$ |
| $c_1 = \frac{E}{a}$ | $c_{n+1} = \frac{E - a^n L}{a^{n+1} - a^n}$ |
| $c_j = 0$ for $\forall j \neq 0, 1$ | $c_j = 0$ for $\forall j \neq n, n+1$ |
| $\frac{r}{w} = \frac{\lambda - 1}{a}$ | $\frac{r}{w} = \frac{\lambda - 1}{a^n(a - \lambda)}$ |
| $C = L + (\lambda - 1)\frac{E}{a}$ | $C = \frac{\lambda^{n+1} - \lambda^n}{a^{n+1} - a^n}E + \frac{\lambda^n(a - \lambda)}{a - 1}L$ |
| $\Leftrightarrow E_{0,1} = \frac{a}{\lambda - 1}(C - L)$ | $\Leftrightarrow E_{n,n+1} = \left[C - \frac{\lambda^n(a-\lambda)}{a-1}L\right]\frac{a^{n+1} - a^n}{\lambda^{n+1} - \lambda^n}$ |

[17] This graph may apprear similar to the H-O models with multiple diversification cones in the literature of international trade. However, the mechanisms are different. Leamer (1987) and other papers in this literature mainly consider (small) open economies where the production structure of a country is determined by international specialization and each good has to be consumed. In our closed-economy general equilibrium model, which set of goods should be consumed and produced is an endogenous decision, depending on the domestic demand and endogenous factor prices.

The static equilibrium is summarized verbally in the following proposition.

**Proposition 1** *In a closed economy, generically only two industries with their capital intensities adjacent to the aggregate capital-labor ratio,* $\frac{E}{L}$*, coexist in the equilibrium. As capital per capita increases, every commodity exhibits an inverse-V-shaped life cycle. A good enters the market, prospers (its output increases) and then declines, and finally is fully replaced by another product with a higher capital intensity.*

Table 3 also shows that the aggregate production function ($C$ as a function of $L$ and $E$) has different forms when the endowment structures are different, reflecting the endogenous structural change in the underlying industries. The relative factor price is $\frac{r}{w} = \frac{\lambda-1}{a^n(a-\lambda)}$ when $E \in [a^n L, a^{n+1}L)$, and it declines in a stair-shaped fashion as $E$ increases. This discontinuity results from the Leontief production assumption. Observe that the capital share in the total output is given by

$$\frac{rE}{rE+wL} = \frac{\left(\frac{\lambda-1}{a-1}\right)E}{\frac{\lambda-1}{a-1}E + \frac{a^n(a-\lambda)}{a-1}L} \tag{12}$$

when $E \in [a^n L, a^{n+1}L]$ for any $n \geq 1$. So the capital share monotonically increases with capital within each diversification cone and then suddenly drops to $\frac{(\lambda-1)}{a-1}$ as the economy enters a different diversification cone, but the capital share always falls in the interval $[\frac{\lambda-1}{a-1}, \frac{(\lambda-1)a}{(a-1)\lambda}]$ for any $n \geq 1$. This is consistent with the Kaldor fact that the capital share is fairly stable over time.[18]

Table 3 also shows that the equilibrium industrial output and the industrial employment are not affected by $\lambda$ in this model. The intuition is straightforward: an increase in $\lambda$ raises the relative productivity of good $n+1$ versus good $n$. But their relative price $\frac{p_{n+1}}{p_n}$ is also equal to $\lambda$, thus it also increases by the same amount. These two forces exactly cancel out each other. However, this result is no longer valid when the substitution elasticity between different goods is less than infinity, which we will show in Section 5.

[18] See Barro and Sala-i-Martin (2003) for more discussion on the robustness of the Kaldor facts.

## 4 Dynamic Model

In the previous static model, we take the capital endowment as exogenous. Now we will develop a dynamic model to fully characterize the industrial dynamics along the growth path of the aggregate economy, where the capital changes endogenously over time.

A representative household decides its intertemporal consumption flows and makes the optimal saving and investment decisions, which determine the evolution of the endowment structure and the optimal capital expenditure $E$ at every time point. Therefore the instantaneous market equilibrium can be obtained exactly the same way as in the static model. By the second welfare theorem, we can characterize the competitive equilibrium by resorting to the following social planner problem:

$$\max_{C(t)} \int_0^\infty \frac{C(t)^{1-\sigma}-1}{1-\sigma} e^{-\rho t} dt$$

subject to

$$\dot{K} = \xi K(t) - E(C(t)), \tag{13}$$

$$K(0) = K_0 \text{ is given,}$$

where $\rho$ is the time discount rate and $K(t)$ is the amount of capital *stock* available at the beginning of time $t$. At each time, the old capital can be transformed into new working capital using the standard AK model technology and $\xi$ is the exogenous technology parameter net of the depreciation rate, which captures the effect of learning by doing in the capital goods production or the investment-specific technology progress (Greenwood *et al*, 1997). The new working capital itself cannot be used for consumption, but it can be used either to produce the consumption good or to save/invest. $E(C(t))$ is the total capital *flow* that is used to produce the aggregate consumption $C(t)$. All the consumption goods are non-storable. The total labor endowment $L$ is constant over time.[19] To ensure a positive consumption growth, we assume $\xi - \rho > 0$. To exclude the explosive solution, we also assume $\frac{\xi-\rho}{\sigma}(1-\sigma) < \rho$. Putting them together, we assume

$$0 < \xi - \rho < \sigma\xi. \tag{14}$$

To obtain closed-form solutions, we continue to assume Leontief production functions for the intermediate goods and function (3) for the final good. From Table 3, we know that $E(C)$ is a strictly increasing, continuous, piece-wise linear function of $C$. It is not differentiable at $C = \lambda^i L$, for any $i = 0, 1, \ldots$. Therefore, the above dynamic problem may involve changes in the functional forms of the state equation: (13) can be explicitly rewritten as

$$\dot{K} = \begin{cases} \xi K, & \text{when} \quad C \leq L \\ \xi K - E_{0,1}(C), & \text{when} \quad L \leq C \leq \lambda L \\ \xi K - E_{n,n+1}(C), & \text{when} \quad \lambda^n L \leq C \leq \lambda^{n+1} L, \text{ for } n \geq 1 \end{cases},$$

where $E_{n,n+1}(C)$ is defined in Table 3 for any $n \geq 0$. We can also verify that the objective function is strictly increasing, differentiable and strictly concave while the constraint set forms a continuous convex-valued correspondence, hence the equilibrium must exist and also be unique.

Let $t_0$ denote the *last* time point when the aggregate consumption equals $L$ (that is, only good 0 is produced), and $t_n$ denote the *first* time point when $C = \lambda^n L$ (that is, only good $n$ is produced) for $n \geq 1$. As can be verified later, the aggregate consumption $C$ is monotonically increasing over time in the equilibrium, hence the problem can also be written as

$$\max_{C(t)} \int_0^{t_0} \frac{C(t)^{1-\sigma} - 1}{1-\sigma} e^{-\rho t} dt + \sum_{n=0}^{\infty} \int_{t_n}^{t_{n+1}} \frac{C(t)^{1-\sigma} - 1}{1-\sigma} e^{-\rho t} dt$$

[19] It is straightforward to examine how exogenous changes in the "effective labor" (for example, let $L(t) = L_0 e^{\gamma t}$ for some $\gamma > 0$) may affect economic dynamics.

subject to

$$\dot{K} = \begin{cases} \xi K & \text{when} \quad 0 \le t \le t_0 \\ \xi K - E_{0,1}(C), & \text{when} \quad t_0 \le t \le t_1 \\ \xi K - E_{n,n+1}(C), & \text{when} \quad t_n \le t \le t_{n+1}, \text{ for } n \ge 1 \end{cases},$$

$K(0) = K_0$ is given,

where $t_n$ is to be endogenously determined for any $n \ge 0$.

Table 3 indicates that goods 0 and 1 are produced during the time period $[t_0, t_1]$ and $E(C) = E_{0,1}(C) \equiv \frac{a}{\lambda-1}(C-L)$. When $t_n \le t \le t_{n+1}$ for $n \ge 1$, goods $n$ and $n+1$ are produced. Correspondingly, $E(C) = E_{n,n+1}(C) \equiv \left[C - \frac{\lambda^n(a-\lambda)}{a-1}L\right] \frac{a^{n+1}-a^n}{\lambda^{n+1}-\lambda^n}$. If $K_0$ is sufficiently small (this will be more precise soon), then there exists a time period $[0, t_0]$ in which only good 0 is produced and all the working capital is saved for the future, so that $E = 0$ when $0 \le t \le t_0$. If $K_0$ is large, on the other hand, the economy may start with producing good $h$ and $h+1$ for some $h \ge 1$, so $t_0 = t_1 = \cdots = t_h = 0$ in the equilibrium.

To solve the above dynamic problem, following Kamien and Schwartz (1991), we set the *discounted-value* Hamiltonian in the interval $t_n \le t \le t_{n+1}$, and use subscripts "$n, n+1$" to denote all the variables during this time interval:

$$\begin{aligned} H_{n,n+1} &= \frac{C(t)^{1-\sigma} - 1}{1-\sigma} e^{-\rho t} + \eta_{n,n+1} [\xi K(t) - E_{n,n+1}(C(t))] \\ &\quad + \zeta_{n,n+1}^{n+1}(\lambda^{n+1}L - C(t)) + \zeta_{n,n+1}^{n}(C(t) - \lambda^n L) \end{aligned} \tag{15}$$

where $\eta_{n,n+1}$ is the co-state variable, $\zeta_{n,n+1}^{n+1}$ and $\zeta_{n,n+1}^{n}$ are the Lagrangian multipliers for the two constraints $\lambda^{n+1}L - C(t) \ge 0$ and $C(t) - \lambda^n L \ge 0$, respectively. The first-order condition and Kuhn-Tucker conditions are

$$\frac{\partial H_{n,n+1}}{\partial C} = C(t)^{-\sigma} e^{-\rho t} - \eta_{n,n+1} \frac{a^{n+1} - a^n}{\lambda^{n+1} - \lambda^n} - \zeta_{n,n+1}^{n+1} + \zeta_{n,n+1}^{n} = 0, \tag{16}$$

$$\zeta_{n,n+1}^{n+1}(\lambda^{n+1}L - C(t)) = 0,\ \zeta_{n,n+1}^{n+1} \geq 0, \lambda^{n+1}L - C(t) \geq 0,$$
$$\zeta_{n,n+1}^{n}(C(t) - \lambda^{n}L) = 0,\ \zeta_{n,n+1}^{n} \geq 0, C(t) - \lambda^{n}L \geq 0.$$

We also have

$$\eta_{n,n+1}'(t) = -\frac{\partial H_{n,n+1}}{\partial K} = -\eta_{n,n+1}\xi. \tag{17}$$

In particular, when $C(t) \in (\lambda^n L, \lambda^{n+1} L)$, $\zeta_{n,n+1}^{n+1} = \zeta_{n,n+1}^{n} = 0$, and equation (16) becomes

$$C(t)^{-\sigma} e^{-\rho t} = \eta_{n,n+1} \frac{a^{n+1} - a^n}{\lambda^{n+1} - \lambda^n}. \tag{18}$$

The left hand side is the marginal utility gain by increasing one unit of aggregate consumption, while the right hand side is the marginal utility loss due to the decrease in capital because of that additional unit of consumption, which by the chain's rule can be decomposed into two multiplicative terms: the marginal utility of capital $\eta_{n,n+1}$ and the marginal capital requirement for each additional unit of aggregate consumption $\frac{a^{n+1}-a^n}{\lambda^{n+1}-\lambda^n}$ (see Table 3). Taking the logarithm on both sides of equation (18) and differentiating with respect to $t$, we obtain the regular Euler equation:

$$\frac{\overset{\bullet}{C}(t)}{C(t)} = \frac{\xi - \rho}{\sigma}, \tag{19}$$

for $t_n \leq t \leq t_{n+1}$ for any $n \geq 0$. The strictly concave utility function implies that the optimal consumption flow $C(t)$ must be continuous and sufficiently smooth (with no kinks) throughout time; hence from (19) we obtain:

$$C(t) = C(t_0) e^{\frac{\xi-\rho}{\sigma}(t-t_0)} \text{ for any } t \geq t_0 > 0. \tag{20}$$

Following Kamien and Schwartz (1991), we have two additional necessary conditions at $t = t_{n+1}$:

$$H_{n,n+1}(t_{n+1}) = H_{n+1,n+2}(t_{n+1}), \tag{21}$$

$$\eta_{n,n+1}(t_{n+1}) = \eta_{n+1,n+2}(t_{n+1}). \tag{22}$$

Substituting equations (21) and (22) into (15), we can verify that $K^{-}(t_{n+1}) = K^{+}(t_{n+1})$. In other words, $K(t)$ is also continuous. Observe that

$$C(t_0)e^{\frac{\xi-\rho}{\sigma}(t_n-t_0)} = C(t_n) = \lambda^n L \text{ when } t_0 > 0, \tag{23}$$

which implies

$$t_n = \frac{\log \frac{\lambda^n L}{C(t_0)} + \frac{\xi-\rho}{\sigma}t_0}{\frac{\xi-\rho}{\sigma}}, \text{ when } t_0 > 0. \tag{24}$$

Define $m_n \equiv t_{n+1} - t_n$, which measures the length of the time period during which both good $n$ and good $n+1$ are produced (that is, the duration of the diversification cone for good $n$ and good $n+1$). We must have

$$m_n = m \equiv \frac{\sigma \log \lambda}{\xi - \rho}. \tag{25}$$

The comparative statics for equation (25) is summarized in the following proposition.

**Proposition 2** *Whenever $t_i > 0$ for some $i \geq 0$, the duration of each diversification cone for goods $n$ and $n+1$ is identically equal to $m$ for any $n \geq i$. The speed of industrial upgrading (measured by $\frac{1}{m}$) strictly increases with the technological efficiency $\xi$, and intertemporal elasticity of substitution $\frac{1}{\sigma}$, but strictly decreases with the productivity gap $\lambda$ and the time discount rate $\rho$.*

The intuition for the proposition is the following: when the household is more impatient (larger $\rho$), it will consume more and save less and hence the industrial upgrading slows down. When the productivity gap is larger (larger $\lambda$), the marginal product of the current goods is bigger, therefore it pays to stay longer. When the production of the capital good becomes more efficient ($\xi$), capital can be accumulated faster, so the upgrade speed is increased. When the aggregate consumption is more substitutable across time (larger $\frac{1}{\sigma}$), the household is more willing to substitute current consumption for future consumption, which also boosts saving and then causes a quicker industrial upgrading. Observe that the full life of each industry equals $2m$.

We are now ready to derive the industrial dynamics for the entire time period. The industrial dynamics depends on the initial capital stock, $K(0)$. We show in the Appendix that there exists a series of increasing constants, $\vartheta_0, \vartheta_1, \cdots, \vartheta_n, \vartheta_{n+1}, \cdots$, such that if $0 < K(0) \leq \vartheta_0$, the economy will start by producing good 0 only until the capital stock reaches $\vartheta_0$ (Appendix 3 fully characterizes this case); if $\vartheta_n < K(0) \leq \vartheta_{n+1}$, the economy will start by producing goods $n$ and $n+1$ for any $n \geq 0$. Furthermore, we can show that $K(t_n) \equiv \vartheta_n$ for any $K(0) < \vartheta_n$. That is, irrespective of the level of initial capital stock, the economy always starts to produce good $n+1$ whenever its capital stock just reaches $\vartheta_n$.

To be more concrete, consider the case when $\vartheta_0 < K(0) \leq \vartheta_1$, where the threshold values $\vartheta_0$ and $\vartheta_1$ can be explicitly solved (see Appendix 2). That is, the economy will start by producing goods 0 and 1. Using equation (20) and Table 3, we know that when $t \in [0, t_1]$,

$$E(t) = \frac{a}{\lambda - 1}(C(t) - L) = \frac{a}{\lambda - 1}(C(0)e^{\frac{\xi - \rho}{\sigma}t} - L).$$

Correspondingly,

$$\dot{K} = \xi K(t) - \frac{a}{\lambda - 1}(C(0)e^{\frac{\xi - \rho}{\sigma}t} - L).$$

Solving this first-order differential equation with the condition $K(0) = K_0$, we obtain

$$K(t) = \frac{-\frac{aC(0)}{\lambda-1}}{\frac{\xi-\rho}{\sigma} - \xi} e^{\frac{\xi-\rho}{\sigma}t} + \frac{-aL}{\xi(\lambda-1)} + \left[K_0 + \frac{\frac{aC(0)}{\lambda-1}}{\frac{\xi-\rho}{\sigma} - \xi} + \frac{aL}{\xi(\lambda-1)}\right] e^{\xi t},$$

which yields

$$\vartheta_1 \equiv K(t_1) = \frac{-\frac{a\lambda L}{\lambda-1}}{\frac{\xi-\rho}{\sigma} - \xi} + \frac{-aL}{\xi(\lambda-1)} + \left[K_0 + \frac{\frac{aC(0)}{\lambda-1}}{\frac{\xi-\rho}{\sigma} - \xi} + \frac{aL}{\xi(\lambda-1)}\right] \left(\frac{\lambda L}{C(0)}\right)^{\frac{\xi\sigma}{\xi-\rho}}.$$

When $t \in [t_n, t_{n+1}]$, for any $n \geq 1$, the transition equation of capital stock (13) becomes

$$\dot{K} = \xi K(t) - \left[C(0)e^{\frac{\xi-\rho}{\sigma}t} - \frac{\lambda^n(a-\lambda)}{a-1}L\right] \frac{a^{n+1}-a^n}{\lambda^{n+1}-\lambda^n} \text{ when } t \in [t_n, t_{n+1}], \text{ for any } n \geq 1.$$

Solving the above differential equation, we obtain:

$$K(t) = \alpha_n + \beta_n e^{\frac{\xi-\rho}{\sigma}t} + \gamma_n e^{\xi t} \text{ when } t \in [t_n, t_{n+1}], \text{ for any } n \geq 1 \tag{26}$$

where

$$\begin{aligned}
\alpha_n &= -\left(\frac{a^{n+1}-a^n}{\lambda^{n+1}-\lambda^n}\right) \frac{\lambda^n(a-\lambda)L}{\xi(a-1)}, \\
\beta_n &= -\left(\frac{a^{n+1}-a^n}{\lambda^{n+1}-\lambda^n}\right) \frac{C(0)}{\left(\frac{\xi-\rho}{\sigma} - \xi\right)}, \\
\gamma_n &= \left[\frac{\lambda^n L}{C(0)}\right]^{\frac{-\xi\sigma}{\xi-\rho}} \left\{\vartheta_n + \frac{\left(a^{n+1}-a^n\right)L}{\lambda-1}\left[\frac{1}{\left(\frac{\xi-\rho}{\sigma} - \xi\right)} + \frac{(a-\lambda)}{\xi(a-1)}\right]\right\}.
\end{aligned}$$

Again the endogenous change in the functional form of the capital accumulation path (26) reflects the structural changes that underlie the aggregate economic growth. Note that $\{\vartheta_n\}_{n=2}^{\infty}$ are all constants, which can be sequentially pinned down: $\vartheta_n \equiv K(t_n)$ can be computed from equation (26) with $K(t_{n-1})$ known.

For each individual industry, using equation (20) and Table 3, we obtain

$$c_n^*(t) = \begin{cases} \frac{C(0)e^{\frac{\xi-\rho}{\sigma}t}}{\lambda^n-\lambda^{n-1}} - \frac{L}{\lambda-1} & \text{when} \quad t \in [t_{n-1}, t_n] \\ -\frac{C(0)e^{\frac{\xi-\rho}{\sigma}t}}{\lambda^{n+1}-\lambda^n} + \frac{\lambda L}{\lambda-1}, & \text{when} \quad t \in [t_n, t_{n+1}] \\ 0, & \text{otherwise} \end{cases}, \text{ for all } n \geq 2$$

$$c_1^*(t) = \begin{cases} \frac{C(0)e^{\frac{\xi-\rho}{\sigma}t}-L}{\lambda-1}, & \text{when} \quad t \in [0, t_1] \\ -\frac{C(0)e^{\frac{\xi-\rho}{\sigma}t}}{\lambda^2-\lambda} + \frac{\lambda L}{\lambda-1}, & \text{when} \quad t \in [t_1, t_2] \\ 0, & \text{otherwise} \end{cases},$$

$$c_0^*(t) = \begin{cases} L - \frac{C(0)e^{\frac{\xi-\rho}{\sigma}t}-L}{\lambda-1}, & \text{when} \quad t \in [0, t_1] \\ 0, & \text{otherwise} \end{cases},$$

where $C(0)$ can be uniquely determined by using the transversality condition (see Appendix 2) and the endogenous time points $t_n$ given by (24) for any $n \geq 1$. Recall $t_0 = 0$ in this case. The above mathematical equations fully characterize the industrial dynamics for each industry over the whole life cycle while the aggregate consumption growth is still given by (19). These mathematical equations can be read as follows:

**Proposition 3** *There exist a strictly increasing and non-negative sequence of endogenous threshold values for capital stock, $\{\vartheta_i\}_{i=0}^{\infty}$, which are all independent of the initial capital stock $K(0)$, such that the economy starts to produce good $n$ when its capital stock $K(t)$ reaches $\vartheta_{n-1}$. $K(t)$ evolves following the equation (26), while the total consumption $C(t)$ remains constant at $L$ until $t_0$, after which it grows exponentially at the constant rate $\frac{\xi-\rho}{\sigma}$. The output of each industry evolves in an inverse-V-shaped pattern: When capital stock $K(t)$ reaches $\vartheta_{n-1}$, good $n$ enters the market and its output grows approximately at the constant rate $\frac{\xi-\rho}{\sigma}$ until capital stock $K(t)$ reaches $\vartheta_n$; its output then declines approximately at the constant rate $\frac{\xi-\rho}{\sigma}$, and exits from the market at the time when $K(t)$ reaches $\vartheta_{n+1}$.*[20]

[20] More rigorously, the sum of the good $n$'s output and a constant, $c_n^*(t) + \frac{L}{\lambda-1}$, grows at the

Proposition 3 can be illustrated in the inverse-V-shaped pattern of industrial dynamics depicted in Figure 6.

[Figure 6]

It is clear that our theoretical predictions of industrial dynamics are consistent with Facts 2, 3, and 4, while at the same time our theoretical results are also consistent with the Kaldor facts that the growth rate of total consumption remains constant and the capital-output ratio is approximately stable, as shown in equation (12). If the initial capital stock is sufficiently small (that is, $K_0 < \vartheta_0$ as characterized in Appendix 3), then the economy will first have a constant output level equal to $L$ (Malthusian growth) until the capital stock $K(t) = \vartheta_0$, which occurs at $t_0 > 0$, after which the aggregate consumption growth rate permanently changes to $\frac{\xi-\rho}{\sigma}$ (Solow growth).

## 5 A More General Setup

One salient counterfactual feature in the model equilibrium is that at any time point at most two industries can coexist because we assume that the CES substitution elasticity across different industries is infinity. In this section we show that all the industries will coexist when the substitution elasticity is finite, but the key results remain valid, namely, each industry still demonstrates an inverse-V-shaped pattern of industrial dynamics with more capital intensive industries reaching the peak later, although no explicit solutions can be obtained due to the "curse of dimensionality" as no industries exit any more.

To see the results, assume everything else remains the same except that now (3) is changed to a more general CES function:

$$C = \left[\sum_{n=0}^{\infty} \lambda_n c_n^{\gamma}\right]^{\frac{1}{\gamma}}, \tag{27}$$

constant rate $\frac{\xi-\rho}{\sigma}$, and then $c_n^*(t) - \frac{\lambda L}{\lambda-1}$ declines at the constant rate $\frac{\xi-\rho}{\sigma}$.

where $\gamma \leq 1$. The previous sections have provided a full characterization for the special case when $\gamma = 1$, so now we will focus on the case when $\gamma < 1$. The imperfect substitution elasticity implies that every good $n \geq 0$ will be produced in the equilibrium. For any $n \geq 1$, the MRT is equal to the price ratio:

$$MRT_{n+1,n} = \lambda \left( \frac{c_{n+1}}{c_n} \right)^{\gamma - 1} = \frac{p_{n+1}}{p_n} = \frac{w + a^{n+1} r}{w + a^n r},$$

which implies that

$$\frac{c_{n+1}}{c_n} = \lambda^{\frac{1}{1-\gamma}} \left( \frac{1 + a^{n+1}\theta}{1 + a^n \theta} \right)^{\frac{1}{\gamma - 1}}, \forall n \geq 1 \tag{28}$$

where $\theta$ denotes the rental wage ratio $\frac{r}{w}$. Thus

$$c_n = c_1 \left(1 + a\theta\right)^{\frac{1}{1-\gamma}} \left( \frac{\lambda^{n-1}}{1 + a^n \theta} \right)^{\frac{1}{1-\gamma}}, \forall n \geq 1. \tag{29}$$

In particular, we have

$$\frac{c_1}{c_0} = \lambda^{\frac{1}{1-\gamma}} \left(1 + a\theta\right)^{\frac{1}{\gamma - 1}}. \tag{30}$$

Combining (29) and (30), we can express the output of each good as a function of $c_1$ and $\theta$. Using the market clearing conditions for labor and capital, we obtain

$$L = c_1 \left[ \left( \frac{1 + a\theta}{\lambda} \right)^{\frac{1}{1-\gamma}} + \sum_{n=1}^{\infty} \lambda^{\frac{n-1}{1-\gamma}} \left( \frac{1 + a^n \theta}{1 + a\theta} \right)^{\frac{1}{\gamma - 1}} \right], \tag{31}$$

$$E = c_1 \sum_{n=1}^{\infty} a^n \lambda^{\frac{n-1}{1-\gamma}} \left( \frac{1 + a^n \theta}{1 + a\theta} \right)^{\frac{1}{\gamma - 1}}. \tag{32}$$

Thus the factor price ratio $\theta$ is determined by

$$\frac{E}{L} = \frac{\sum_{n=1}^{\infty} a^n \lambda^{\frac{n-1}{1-\gamma}} \left(1 + a^n \theta\right)^{\frac{1}{\gamma - 1}}}{\left(\frac{1}{\lambda}\right)^{\frac{1}{1-\gamma}} + \sum_{n=1}^{\infty} \lambda^{\frac{n-1}{1-\gamma}} \left(1 + a^n \theta\right)^{\frac{1}{\gamma - 1}}}. \tag{33}$$

Recall that we assume $\lambda < a$ (condition (7)) to rule out the trivial case in the static model. Here we strengthen this assumption by assuming $\lambda < a^{\gamma}$ to ensure the right-hand side of (33) is well defined (finite).

Next we offer a series of lemmas, which lead us to a *Generalized Rybzynski Theorem.* Lemma 1 shows that the rental-wage ratio $\theta$ decreases as the capital-labor ratio $\frac{E}{L}$ increases.

**Lemma 1** *$\theta$ and output $c_n$ for any $n \geq 0$ can all be uniquely determined. In addition, $\theta$ strictly decreases with $\frac{E}{L}$.*

**Proof.** See Appendix 4 ■

Lemma 2 establishes that when $E$ increases, holding $L$ constant, the equilibrium output of a higher-indexed industry increases disproportionately more (or decreases disproportionately less) than any lower-indexed industry. So when $E$ increases, if $c_n$ increases, then $c_{n+h}$ increases for all $h \geq 1$ and if $c_n$ decreases, then $c_{n-h}$ decreases for all $h \geq 1$.

**Lemma 2** *$\frac{c'_{n+1}(E)}{c_{n+1}(E)} > \frac{c'_n(E)}{c_n(E)}$ for any $n \geq 0$ and for any $E$.*

**Proof.** See Appendix 4 ■

The following lemma shows that for any given $E$ there exists a threshold industry $n^*$ such that all the industries that are more capital intensive than industry $n^*$ will expand when $E$ increases marginally, while industry $n^*$ and all the industries that are less capital intensive than $n^*$ will marginally shrink. Furthermore, if the single-peaked condition is satisfied (specified in the Appendix 4), the index of the threshold industry $n^*$ weakly increases as $E$ increases. That is, the threshold industry itself shifts toward the more capital intensive direction as $E$ increases.

**Lemma 3** *(A) $\frac{\partial c_0}{\partial E} < 0$. Moreover, for any given $E$, there must exist a unique finite positive integer $n^*(E)$ such that $\frac{\partial c_n}{\partial E} \geq 0$ holds if and only if $n > n^*(E)$. (B)*

*Each industry will eventually decline as $E$ becomes sufficiently large. (C) If the single-peaked condition is satisfied, $n^*(E)$ weakly increases in $E$.*

**Proof.** See Appendix 4 ■

Now we combine all the previous lemmas together. For any industry $n \leq n^*(E)$ for some given $E$, the industry always declines when $E$ increases. For any industry $n > n^*(E')$ for some $E'$, it expands as $E'$ increases up to a point $E''$ such that $n < n^*(E'')$. Industry $n$ will then decline with $E$ after $E$ surpasses $E''$. Therefore, the industry exhibits an inverse-V-shaped pattern as the economy becomes more capital abundant. To summarize, we have:

**Proposition 4** *(Generalized Rybzynski Theorem): Suppose the production function is CES as defined in (27). All the industries will coexist in the equilibrium. In addition, when capital endowment $E$ increases, the output and the employment in industry $n$ (for any $n \geq 1$) will both exhibit an inverse-V-shaped pattern: an industry first expands with $E$ and then declines with $E$ after the industrial output reaches its peak. The more capital intensive an industry, the larger the capital endowment at which the industry reaches its peak.*

For illustrative purpose, Figure 7 plots how the industrial output of the first six industries ($n = 0, 1, 2.., 5$) changes as the capital endowment $E$ becomes larger. The output of industry 0 $c_0$ becomes smaller when $E$ increases, as predicted by Lemma 3. All the other industries demonstrate a hump-shape pattern. The more capital intensive an industry, the "later" it reaches its peak, as predicted by the above proposition.

[Insert Figure 7]

Recall the Rybczynski Theorem states that in a static model with two goods and two factors, the capital-intensive sector expands while the labor-intensive sector shrinks as the economy becomes more capital abundant. However, it is silent about what happens when the number of goods exceeds two. In a model with $N$ ($N > 2$) goods and two factors, as the country becomes more capital abundant, the existing theory only suggests that there must be some sector whose output rises and some other sector whose output falls, but it is not clear which sectors expand and which sectors shrink (Feenstra, 2004). Now Proposition 4 generalizes the Rybczynski Theorem by extending the commodity space from two-dimensional to infinite dimensional, so we call this proposition the Generalized Rybzynski Theorem.

We can show that similar patterns also obtain even if all the industries have the same marginal productivity in producing the final goods ($\lambda_n = 1$ for all $n$), as illustrated by Figure 8. It implies that the key mechanism that drives the industrial upgrading is the change in the factor endowment.

[Insert Figure 8]

For the dynamic analysis, first observe that

$$\frac{E}{C} = \frac{\sum_{n=1}^{\infty} \frac{a^n}{1+a^n\theta} \lambda^{\frac{n}{1-\gamma}} (1 + a^n\theta)^{\frac{\gamma}{\gamma-1}}}{\left[1 + \sum_{n=1}^{\infty} \lambda^{\frac{n}{1-\gamma}} (1 + a^n\theta)^{\frac{\gamma}{\gamma-1}}\right]^{\frac{1}{\gamma}}},$$

where $\theta$ is a strictly decreasing function of $E$, as was proved before. So the above function implicitly determines $E \equiv G(C)$, where function $G(C)$ must satisfy $G'(C) > 0$ and $G''(C) \geq 0$. This is because the marginal productivity of capital in producing the aggregate consumption must be diminishing since the total labor supply is fixed while the aggregate production function is homogeneous of degree one with respect to capital and labor. Define $\psi(C) \equiv \frac{CG''(C)}{G'(C)}$, which is the elasticity of the change in marginal capital expenditure relative to the aggregate consumption change. For any specific time point $t$ in the dynamic equilibrium, we must have the Euler equation:

$$\frac{\dot{C}(t)}{C(t)} = \frac{\xi - \rho}{\sigma + \psi(C(t))},$$

which would degenerate into (19) when $\psi(C) = 0$, just as in the previous linear case ($\gamma = 1$).[21] This indicates that, as long as $\dot{C}(t) > 0$, $E(t)$ must be strictly increasing over time in the equilibrium (holding $L$ fixed, we still assume $\xi - \rho > 0$ as before). Consequently, the dynamic path of the output of each individual industry must still exhibit an inverse-V-shaped pattern in the full-fledged dynamic model, but it is no longer possible to obtain closed-form solutions for the entire dynamic path.

## 6 Conclusion

In this paper we develop a highly tractable growth model with infinite industries to explore the optimal industrial structure and the dynamics of each industry in a frictionless and deterministic closed economy. The model shows that the optimal industrial structures are endogenously different at different levels of development. Closed-form solutions are obtained to fully characterize the endogenous process of perpetual inverse-V-shaped industrial dynamics, which we observe in the data. At the aggregate level, the model is still consistent with the Kaldor facts. We highlight the improvement in the endowment structure (capital accumulation) as the fundamental driving force behind this continuous structural change.

Continuous technology innovation and structural change are two crucial and integrated aspects of sustained economic development. Most existing growth models postulate the same aggregate production function for countries at different development stages, and thus naturally focus on the technology innovation or TFPs driven by learning by doing or creative destruction. However, the structural side, especially at the highly disaggregated level, is largely ignored by the literature. For example, how to help identify the most suitable industries to develop and support at each different development level, if some industrial policies are socially desirable? As the capital requirements and firm sizes may be different for countries at different levels

[21] If $\lim \psi(C(t)) \to \infty$ as $C(t) \to \infty$, then the asymptotic aggregate consumption growth rate will be zero.

of development, which kind of financial institutions can best serve the corresponding industrial structures? How would openness affect a country's industry upgrading? Given that certain assumptions of the first welfare theorem may not be satisfied in the real world, what will be the optimal industrial, financial, trade, and other macroeconomic policies at different development stages? We hope the model developed in this paper may prove useful to help us think further about all these fundamental issues.

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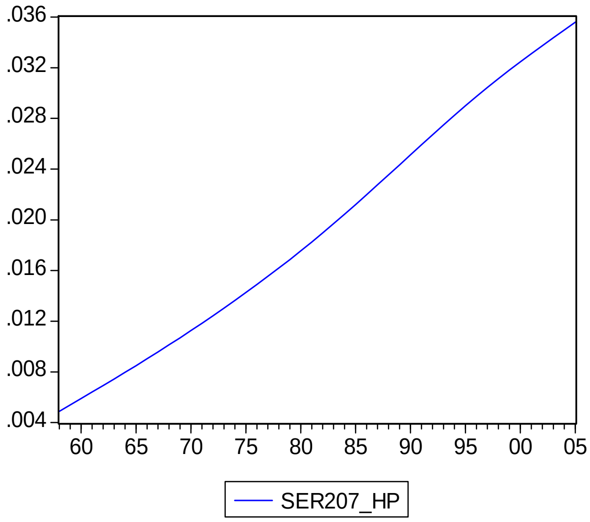



Figure 1A: Monotonic Increase

Industry 326199 (All Other Plastics Product Manufacturing)

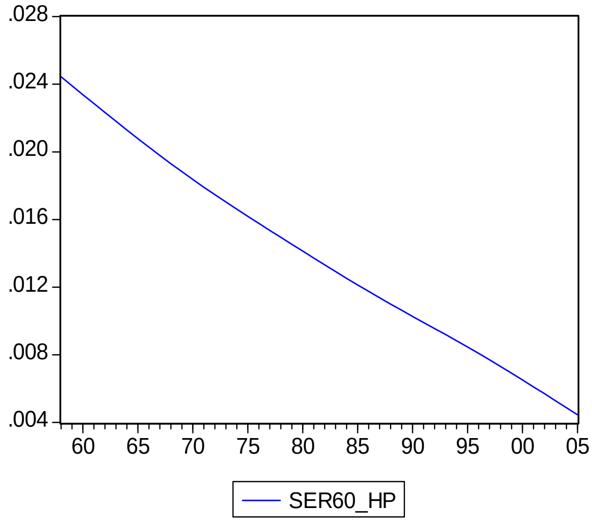



Figure 1B: Monotonic Decrease

Industry 313210 (Broadwoven Fabric Mills)

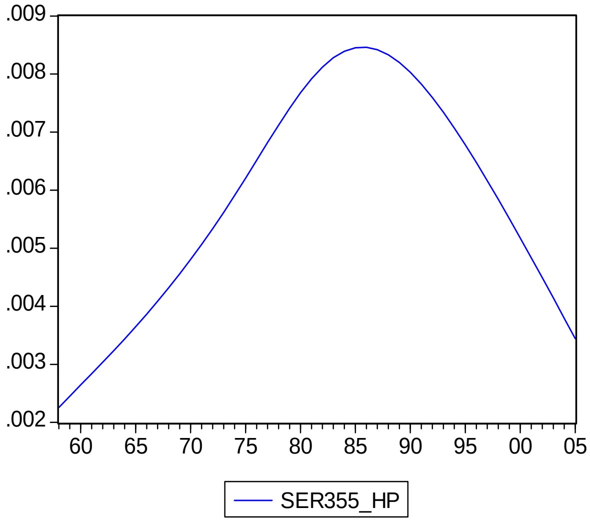



Figure 1C: Inverse V-shape

Industry 334111: Electronic Computer Manufacturing

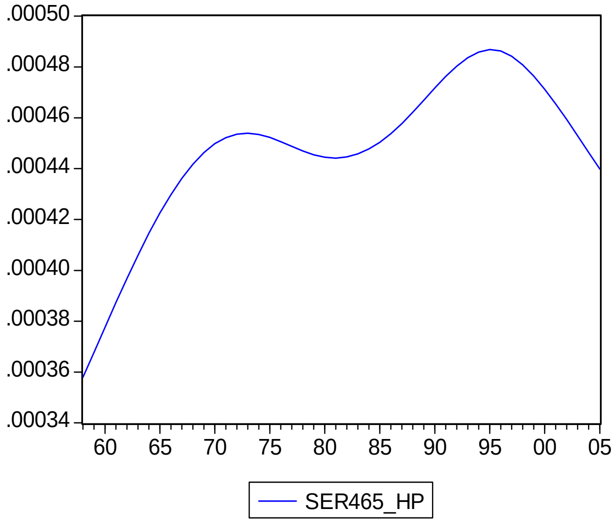



Figure 1D: Multiple Peaks

Industry 339943: Marking Device Manufacturing

Figure 1: Examples of Industrial Dynamics

Note: x-axis is year and y-axis is the labor share (the ratio of sectoral employment to total manufacturing employment after applying HP filter, lambda=1000)

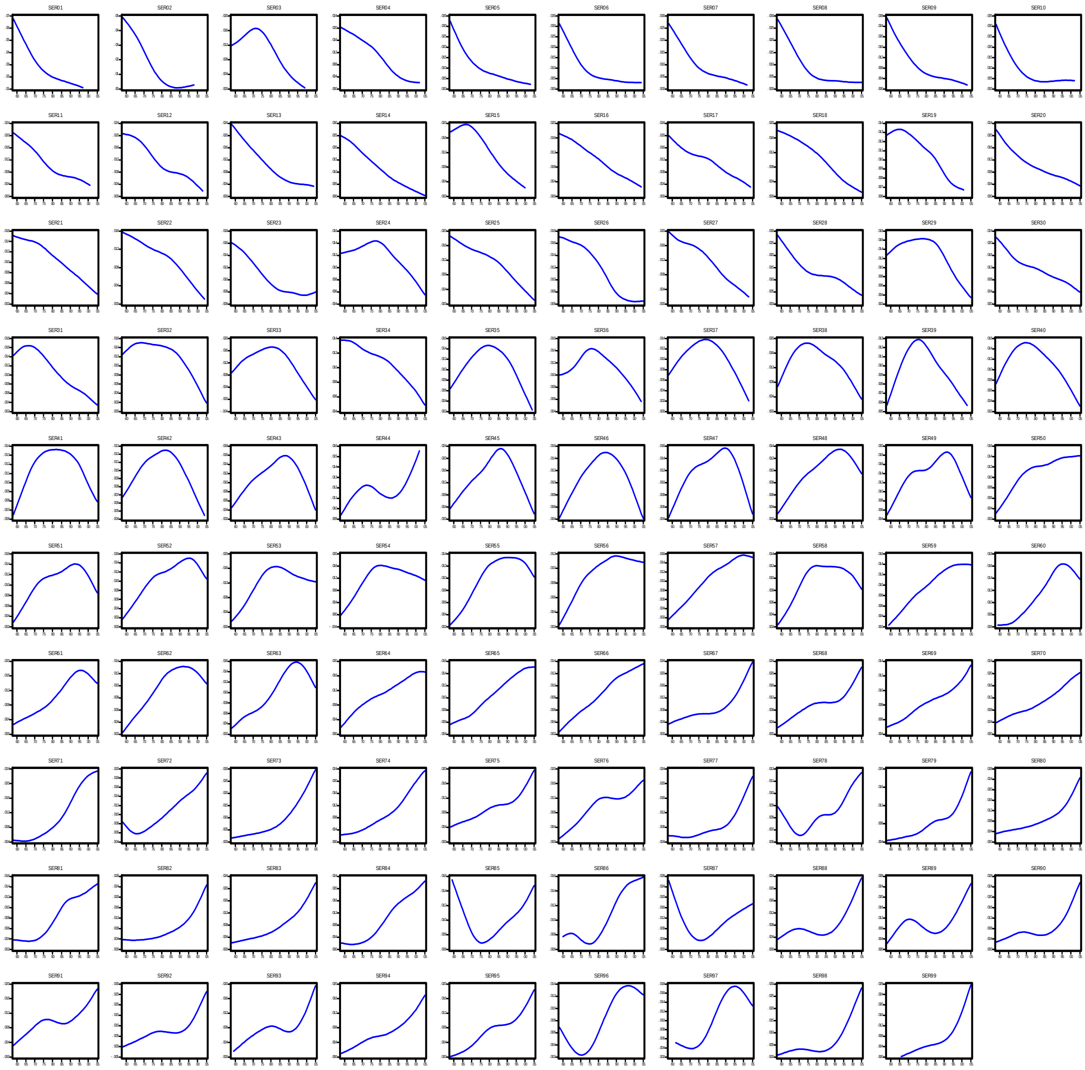
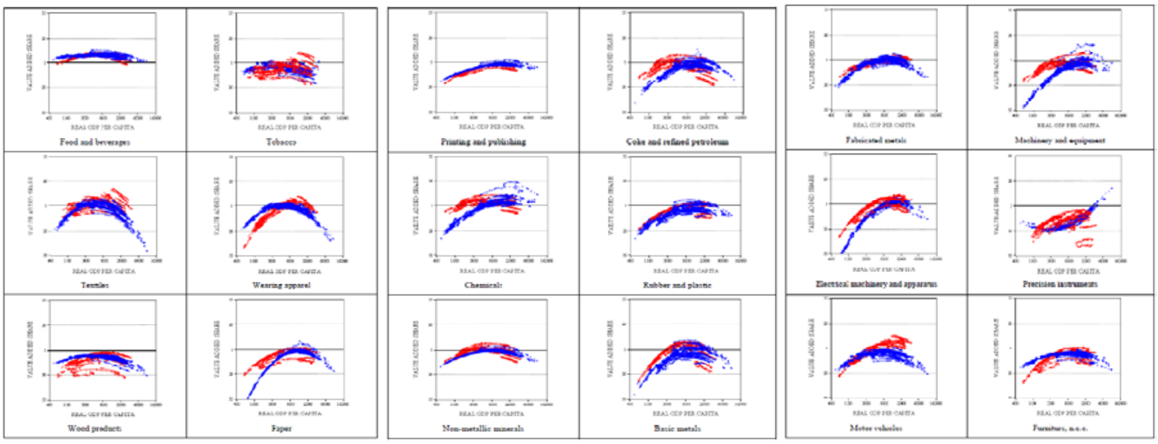


**Figure 2. Industrial Dynamics in US: 1958-2005**

Note: In each box, x-axis is year and y-axis is the labor share. Industries are redefined by the capital-labor ratio, ranked from the most labor intensive industry to the most capital intensive industry.



Panel a: Early Sectors Panel b: Middle Sectors Panel c: Late Sectors

Figure 3. Industrial Value-add Share versus Real GDP per capita (UNIDO, 1963-2006)

Source: This figure is taken from Haraguchi and Rezonja (2010).

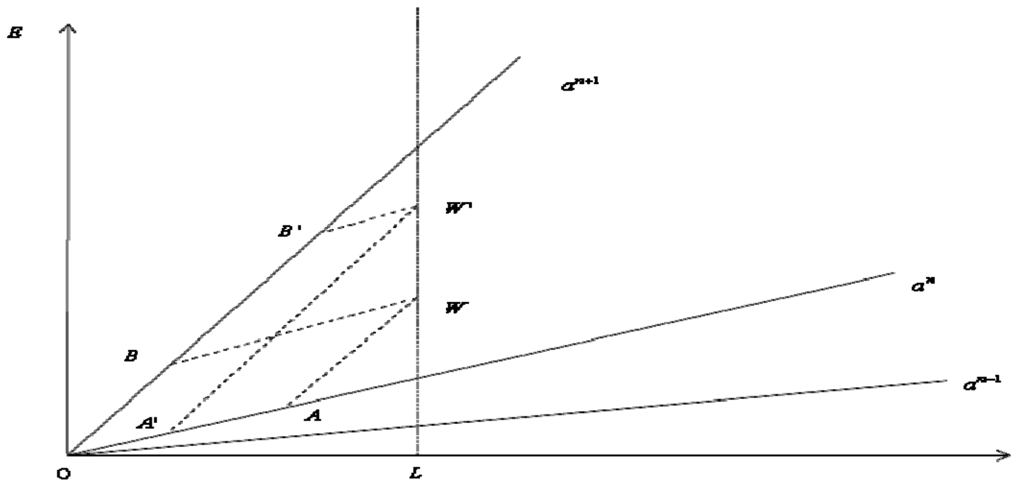



Figure 4. How Endowment Structure Determines Equilibrium Industries

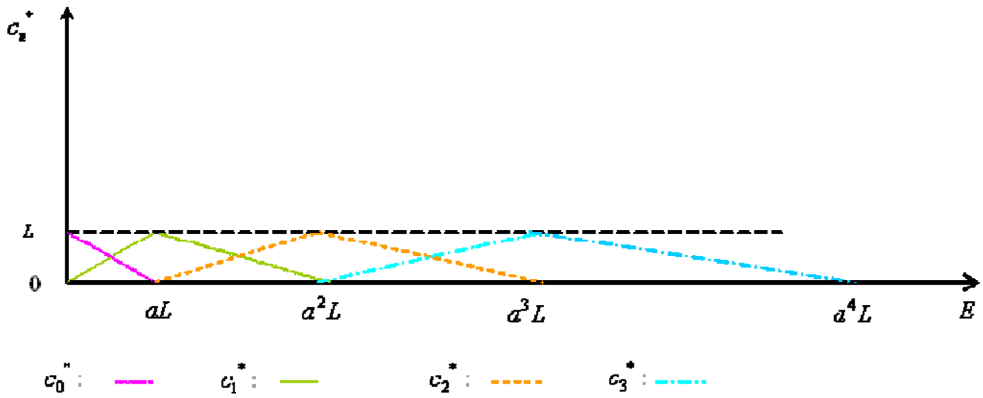



Figure 5. How Industrial Output Changes with Capital Endowment

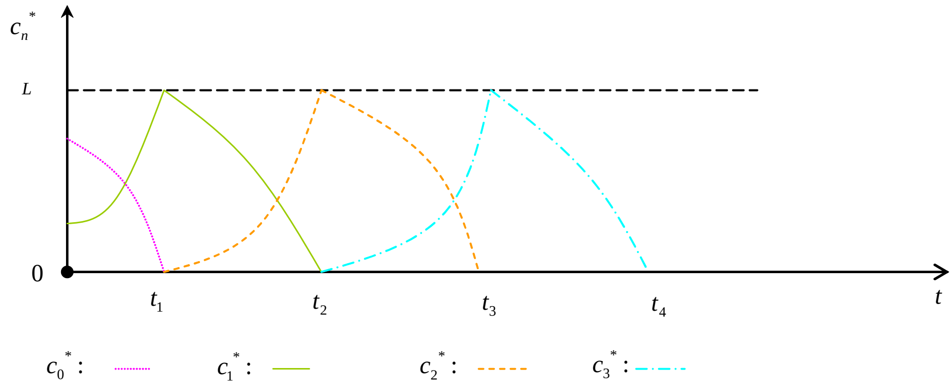



Figure 6. How Industries Evolve over Time when $K_0 \in (\vartheta_0, \vartheta_1)$ .

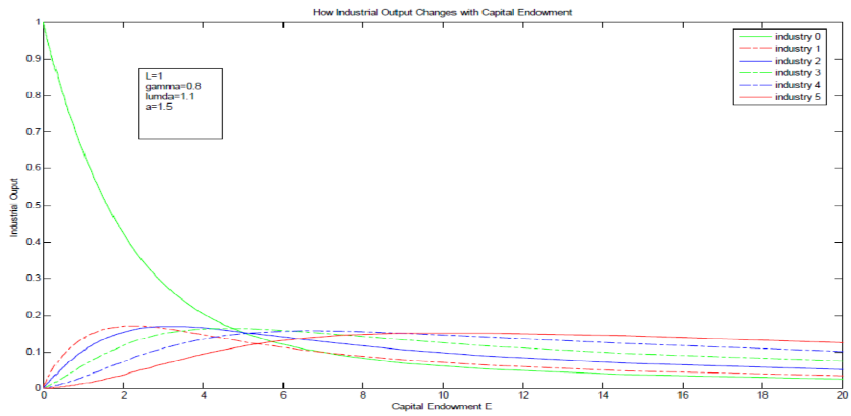



Figure 7. How Industrial Output Changes with Capital Endowment under General CES function when $\lambda > 1$

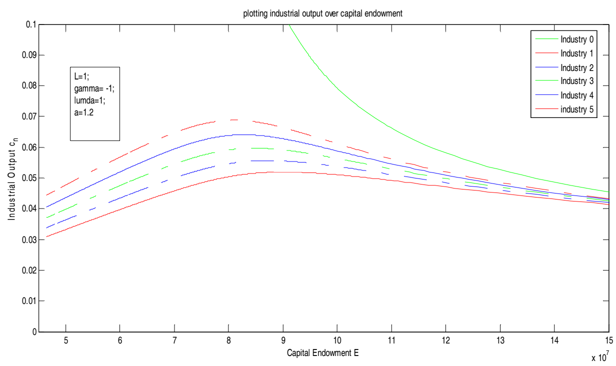



Figure 8. How Industrial Output Changes with Capital Endowment under General CES function when $\lambda = 1$

Table 1: Determinants of the Industry's Share

| Independent variable | Employment share | | Value-added share | |
|---|---|---|---|---|
| | (1) | (2) | (3) | (4) |
| Coherence term 1 | -0.003287*** | | -0.0056138*** | |
| | (0.00021) | | (0.0002155) | |
| Coherence term 2 | | -0.0341269*** | | -0.0425057*** |
| | | (0.0013268) | | (0.0013813) |
| TFP | 0.0023744** | 0.0027773*** | 0.0014396 | 0.0001713 |
| | (0.001159) | (0.0010771) | (0.001188) | (0.0011214) |
| Constant | 0.0211704*** | 0.0329351*** | 0.0166926*** | 0.0350565*** |
| | (0.0019282) | (0.00186) | (0.001977) | (0.0019364) |
| Industry dummies | Yes | Yes | Yes | Yes |
| Observations | 4515 | 4542 | 4515 | 4542 |
| R-squared | 0.1218 | 0.1870 | 0.2240 | 0.2687 |

Note: Coherence term 1 is the absolute value of a normalized difference between sector i's capital-labor ratio and the aggregate capital-labor ratio at year t. Coherence term 2 is the absolute value of a normalized difference between sector i's capital expenditure and the average capital expenditure in all manufacturing sectors at year t. ***indicates significant at 1% level. **indicates significant at 5% level. In addition, we run a Hausman test to determine whether our model specification should include fixed effects. The test results show that the fixed effect model is statistically preferred over the random effect model.

Table 2: Peak Time of Industries

| Independent variable | Peak time of employment share | | Peak time of value-added share | |
|---|---|---|---|---|
| | (1) | (2) | (3) | (4) |
| Capital-labor ratio | 0.535*** | | 0.553*** | |
| | (0.070) | | (0.070) | |
| Capital expenditure | | 123.298*** | | 97.863*** |
| | | (16.550) | | (9.841) |
| Constant | 1958.309*** | 1915.588*** | 1955.004*** | 1924.355*** |
| | (3.371) | (8.876) | (3.625) | (5.525) |
| Observations | 33 | 40 | 34 | 43 |
| R-squared | 0.651 | 0.594 | 0.661 | 0.707 |

Note: We drop industries monotonically declining (increasing) during entire time period, and a few industries which are not single peaked. ***indicates significant at 1% level.

# 7 Appendices (For Online Publication)

## 7.1 Appendix 1

In Appendix 1, we show that in the competitive equilibrium, if two different industrial goods are produced simultaneously, these two goods have to be adjacent in the capital intensities.

**Proof:** First set up the Lagrangian for the household problem with the multiplier denoted by $\mu$, and we obtain the following optimality condition for consumptions:

$$\lambda^n \left( \sum_{n=1}^{\infty} \lambda^n c_n + c_0 \right)^{-\sigma} \leq \mu p_n, \text{ for } \forall n \geq 0, \quad \text{``} = \text{''} \text{ when } c_n > 0. \tag{34}$$

By contradiction, suppose the statement is not true, then there can exist some good $j$ and good $m$ such that $1 \leq j \leq m-2$ , and $c_j$ and $c_m$ are both strictly positive in the equilibrium. Then

$$MRT_{m,j} = \lambda^{m-j} = \frac{a^m r + w}{a^j r + w},$$

which means

$$\frac{r}{w} = \frac{\lambda^{m-j} - 1}{a^j (a^{m-j} - \lambda^{m-j})}.$$

Now we show that at this relative price, good $j+1$ strictly dominates good $j$, that is

$$MRT_{j+1,j} - \frac{a^{j+1} r + w}{a^j r + w} = \lambda - \frac{a^{j+1} \frac{\lambda^{m-j}-1}{a^j (a^{m-j} - \lambda^{m-j})} + 1}{a^j \frac{\lambda^{m-j}-1}{a^j (a^{m-j} - \lambda^{m-j})} + 1} > 0,$$

which is equivalent to

$$\frac{\lambda^n - 1}{a^n - \lambda^n} < \frac{1-\lambda}{\lambda - a}, \text{ for some integer } n \equiv m - j \geq 2. \tag{35}$$

This is true because

$$\begin{aligned} \frac{\lambda^n - 1}{a^n - \lambda^n} &= \frac{(\lambda - 1) \sum_{i=0}^{n-1} \lambda^i}{\lambda^n \left( \left( \frac{a}{\lambda} \right)^n - 1 \right)} = \frac{(\lambda - 1) \sum_{i=0}^{n-1} \lambda^i}{\lambda^n \left( \left( \frac{a}{\lambda} \right) - 1 \right) \sum_{i=0}^{n-1} \left( \frac{a}{\lambda} \right)^i} \\ &= \frac{(\lambda - 1) \sum_{i=0}^{n-1} \lambda^i}{\lambda^{n-1} (a - \lambda) \sum_{i=0}^{n-1} \left( \frac{a}{\lambda} \right)^i} = \frac{(\lambda - 1) \sum_{i=0}^{n-1} \lambda^i}{(a - \lambda) \sum_{i=0}^{n-1} \lambda^i a^{n-1-i}} < \frac{\lambda - 1}{a - \lambda} \text{for any } n \geq 2. \end{aligned}$$

Therefore, it contradicts that good $j$ is produced and consumed. It's straightforward to show that it is possible to have both good 0 and good 1 under some conditions.

Now we need to show that if $c_n > 0$ for some $n \geq 2$ and $c_j = 0$ for any $j \geq 2$ and $j \neq n$, then $c_0 = 0$. From (34), the household would strictly prefer good $n$ to good $n+1$ if

$$\frac{\lambda_{n+1}}{\lambda_n} < \frac{p_{n+1}}{p_n} = \frac{w + a_{n+1}r}{w + a_n r}, \text{for } n = 1, ..$$

or equivalently, $\frac{r}{w} > \frac{\lambda-1}{a^{n+1}-\lambda a^n}$, and good $n$ is strictly preferred to good $n-1$ if and only if $\frac{r}{w} < \frac{\lambda-1}{a^n-\lambda a^{n-1}}$. This means

$$\frac{\lambda - 1}{a^{n+1} - \lambda a^n} < \frac{r}{w} < \frac{\lambda - 1}{a^n - \lambda a^{n-1}}. \tag{36}$$

By contradiction, suppose $c_0 > 0$, then $\lambda_n = \frac{w+a_n r}{w}$ because of (34), that is $\frac{\lambda^n-1}{a^n} = \frac{r}{w}$. So we must have

$$\frac{\lambda - 1}{a^{n+1} - \lambda a^n} < \frac{\lambda^n - 1}{a^n} < \frac{\lambda - 1}{a^n - \lambda a^{n-1}},$$

where the second inequality is equivalent to $\frac{\lambda^n-1}{\lambda-1} < \frac{a}{a-\lambda}$. However, since $a - 1 > \lambda$, there will exist no integer $n \geq 2$ that can satisfy the above inequality because the left hand side is no smaller than $\lambda + 1$. Therefore we must have $c_0 = 0$.

### 7.2 Appendix 2

In Appendix 2, we solve for the initial value of total consumption $C(0)$ when $\vartheta_0 < K(0) \leq \vartheta_1$, and also show how to derive the threshold values for $\vartheta_i, i = 0, 1, 2, ....$ The transversality condition is derived from

$$\lim_{t\to\infty} H(t) = 0,$$

so

$$\lim_{t\to\infty} \left[ \frac{C(t)^{1-\sigma} - 1}{1-\sigma} e^{-\rho t} + \eta_{n(t),n(t)+1} \left[ \xi K(t) - E_{n(t),n(t)+1}(C(t)) \right] \right] = 0.$$

Note that

$$\begin{aligned}
&\lim_{t\to\infty} \left[ \frac{C(t)^{1-\sigma} - 1}{1-\sigma} e^{-\rho t} + \eta_{n(t),n(t)+1} \left[ \xi K(t) - E_{n(t),n(t)+1}(C(t)) \right] \right] \\
&= \lim_{t\to\infty} \left[ \frac{C(0)^{1-\sigma} e^{\frac{\xi-\rho}{\sigma}(1-\sigma)t}}{1-\sigma} e^{-\rho t} + \eta_{n(t),n(t)+1} \left[ \xi K(t) - E_{n(t),n(t)+1}(C(t)) \right] \right] \\
&= \lim_{t\to\infty} \left[ \eta_{n(t),n(t)+1} \left[ \xi K(t) - E_{n(t),n(t)+1}(C(t)) \right] \right] \\
&= \lim_{t\to\infty} \left\{ \eta_{(0)} e^{-\xi t} \left[ \xi K(t) - \left[ C(0) e^{\frac{\xi-\rho}{\sigma} t} - \frac{\lambda^{n(t)}(a-\lambda)}{a-1} L \right] \frac{a^{n(t)+1} - a^{n(t)}}{\lambda^{n(t)+1} - \lambda^{n(t)}} \right] \right\} \\
&= \lim_{t\to\infty} \left\{ \eta_{(0)} \left[ \xi K(t) e^{-\xi t} - \left[ -\frac{e^{-\xi t}(a-\lambda)}{a-1} L \right] \frac{a^{n(t)+1} - a^{n(t)}}{\lambda - 1} \right] \right\} \\
&= \lim_{t\to\infty} K(t) e^{-\xi t},
\end{aligned}$$

thus we must have $\lim_{t\to\infty} K(t)e^{-\xi t} = 0$. When $t \in [0, t_1]$,

$$E(t) = \frac{a}{\lambda - 1}(C(t) - L) = \frac{a}{\lambda - 1}(C(0)e^{\frac{\xi-\rho}{\sigma}t} - L),$$

Correspondingly,

$$\dot{K} = \xi K(t) - E(C(t)) = \xi K(t) - \frac{a}{\lambda - 1}(C(0)e^{\frac{\xi-\rho}{\sigma}t} - L)$$

Solving this first-order differential equation with the condition $K(0) = K_0$, we obtain

$$K(t) = \frac{-\frac{aC(0)}{\lambda-1}}{\frac{\xi-\rho}{\sigma} - \xi}e^{\frac{\xi-\rho}{\sigma}t} + \frac{-aL}{\xi(\lambda - 1)} + \left[K_0 + \frac{\frac{aC(0)}{\lambda-1}}{\frac{\xi-\rho}{\sigma} - \xi} + \frac{aL}{\xi(\lambda - 1)}\right]e^{\xi t}, \quad (37)$$

from which we obtain

$$K(t_1) = \frac{-\frac{a\lambda L}{\lambda-1}}{\frac{\xi-\rho}{\sigma} - \xi} + \frac{-aL}{\xi(\lambda - 1)} + \left[K_0 + \frac{\frac{aC(0)}{\lambda-1}}{\frac{\xi-\rho}{\sigma} - \xi} + \frac{aL}{\xi(\lambda - 1)}\right]\left(\frac{\lambda L}{C(0)}\right)^{\frac{\xi\sigma}{\xi-\rho}}. \quad (38)$$

When $t \in [t_n, t_{n+1}]$ for $\forall n \geq 1$, we have

$$K(t) = -\frac{a^{n+1} - a^n}{\lambda^{n+1} - \lambda^n}\left[\frac{C(0)e^{\frac{\xi-\rho}{\sigma}t}}{\left(\frac{\xi-\rho}{\sigma} - \xi\right)} + \frac{\lambda^n(a-\lambda)L}{\xi(a-1)}\right] + \theta_{n.n+1}e^{\xi t}. \quad (39)$$

which gives

$$\theta_{n.n+1} = \left[\frac{\lambda^n L}{C(0)}\right]^{\frac{-\xi\sigma}{\xi-\rho}}\left\{K(t_n) + \frac{a^{n+1} - a^n}{\lambda - 1}L\left[\frac{1}{\left(\frac{\xi-\rho}{\sigma} - \xi\right)} + \frac{(a-\lambda)}{\xi(a-1)}\right]\right\} \quad (40)$$

Therefore, we obtain equation (26). Substituting $t = t_{n+1} = \frac{\log\frac{\lambda^{n+1}L}{C(0)}}{\frac{\xi-\rho}{\sigma}}$ into (26), we obtain

$$K(t_{n+1}) = \lambda^{\frac{\sigma\xi}{\xi-\rho}}K(t_n) + \frac{a^{n+1} - a^n}{\lambda - 1}L\left[\frac{\lambda^{\frac{\sigma\xi}{\xi-\rho}} - \lambda}{\left(\frac{\xi-\rho}{\sigma} - \xi\right)} + \frac{(a-\lambda)(\lambda^{\frac{\sigma\xi}{\xi-\rho}} - 1)}{\xi(a-1)}\right]$$

Using the recursive induction, we get

$$K(t_n) = \lambda^{\frac{(n-1)\sigma\xi}{\xi-\rho}}K(t_1) + (a-1)B\lambda^{\frac{(n-2)\sigma\xi}{\xi-\rho}}\frac{a\left[1 - \left(a\lambda^{\frac{-\sigma\xi}{\xi-\rho}}\right)^{n-1}\right]}{1 - a\lambda^{\frac{-\sigma\xi}{\xi-\rho}}}, \text{ for any } n \geq 2, \quad (41)$$

where $B$ is a parameter defined as

$$B \equiv \frac{L}{\lambda - 1}\left[\frac{\lambda^{\frac{\sigma\xi}{\xi-\rho}} - \lambda}{\frac{\xi-\rho}{\sigma} - \xi} + \frac{(a-\lambda)\left(\lambda^{\frac{\sigma\xi}{\xi-\rho}} - 1\right)}{\xi(a-1)}\right].$$

The transversality condition $\lim_{t\to\infty} K(t)e^{-\xi t} = 0$ implies that

$$\lambda^{\frac{-\sigma\xi}{\xi-\rho}} K(t_1) + (a-1)B\lambda^{\frac{-2\sigma\xi}{\xi-\rho}} \frac{a}{1 - a\lambda^{\frac{-\sigma\xi}{\xi-\rho}}} = 0,$$

so

$$K(t_1) = -\frac{(a-1)B\lambda^{\frac{-\sigma\xi}{\xi-\rho}} a}{1 - a\lambda^{\frac{-\sigma\xi}{\xi-\rho}}}. \tag{42}$$

To ensure $\vartheta_0 > 0$, we must impose an additional assumption:

$$a < \lambda^{\frac{\sigma\xi}{\xi-\rho}}, \tag{43}$$

which means that capital accumulation speed $\xi$ be sufficiently small relative to the capital intensity parameter $a$, otherwise the economy will never start by only producing good 0 no matter how small $K_0$ is. To ensure $\vartheta_1 \equiv K(t_1) > 0$, we need to further assume

$$\xi > \frac{\rho(a-\lambda)\left(\lambda^{\frac{\sigma\xi}{\xi-\rho}} - 1\right)}{\sigma\left(a - \lambda^{\frac{\sigma\xi}{\xi-\rho}}\right)(1-\lambda) + (a-\lambda)\left(\lambda^{\frac{\sigma\xi}{\xi-\rho}} - 1\right)}. \tag{44}$$

They ensure $B < 0$. Also note that (44) guarantees that $\xi > \rho$. According to (38), we have

$$\left[K_0 + \frac{aC(0)}{(\lambda-1)\left(\frac{\xi-\rho}{\sigma} - \xi\right)} + \frac{aL}{\xi(\lambda-1)}\right]\left(\frac{\lambda L}{C(0)}\right)^{\frac{\xi\sigma}{\xi-\rho}}$$

$$= \frac{a\lambda L}{(\lambda-1)\left(\frac{\xi-\rho}{\sigma} - \xi\right)} + \frac{aL}{\xi(\lambda-1)} - \frac{(a-1)B\lambda^{\frac{-\sigma\xi}{\xi-\rho}} a}{1 - a\lambda^{\frac{-\sigma\xi}{\xi-\rho}}}. \tag{45}$$

We can verify that the right hand side is strictly positive. Observe that the left hand side is a strictly decreasing function of $C(0)$, therefore we can uniquely pin down the optimal $C^*(0)$. (45) immediately implies $\frac{\partial C^*(0)}{\partial K_0} > 0$ and $\frac{\partial C^*(0)}{\partial L} > 0$.

Note that (42) implies that $K(t_1)$ does not depend on $K(0)$, therefore (41) tells that $K(t_n)$ for all $n \geq 1$ are independent from $K(0)$.

To ensure $C^*(0) \le \lambda L$, we need $K_0 \le -\frac{(a-1)B\lambda^{\frac{-\sigma\xi}{\xi-\rho}} a}{1-a\lambda^{\frac{-\sigma\xi}{\xi-\rho}}}$, which requires

$$K_0 \le \vartheta_1 \equiv -\frac{\lambda^{\frac{-\sigma\xi}{\xi-\rho}} a}{1-a\lambda^{\frac{-\sigma\xi}{\xi-\rho}}}\frac{1}{\lambda-1}\left[\frac{\xi\left(a-\lambda^{\frac{\sigma\xi}{\xi-\rho}}\right)(1-\lambda)+\frac{\xi-\rho}{\sigma}(a-\lambda)\left(\lambda^{\frac{\sigma\xi}{\xi-\rho}}-1\right)}{\left(\frac{\xi-\rho}{\sigma}-\xi\right)\xi}\right]L.$$

We also need to ensure $C^*(0) > L$, which requires

$$\left[K_0+\frac{aL}{(\lambda-1)\left(\frac{\xi-\rho}{\sigma}-\xi\right)}+\frac{aL}{\xi(\lambda-1)}\right]\lambda^{\frac{\xi\sigma}{\xi-\rho}} > \frac{a\lambda L}{(\lambda-1)\left(\frac{\xi-\rho}{\sigma}-\xi\right)}+\frac{aL}{\xi(\lambda-1)}-\frac{(a-1)B\lambda^{\frac{-\sigma\xi}{\xi-\rho}} a}{1-a\lambda^{\frac{-\sigma\xi}{\xi-\rho}}},$$

that is

$$K_0 > \vartheta_0 \equiv \frac{a}{(\lambda-1)}\frac{\frac{\xi-\rho}{\sigma}}{\left(\frac{\xi-\rho}{\sigma}-\xi\right)\xi}\left[\frac{\left(1-\lambda^{1-\frac{\sigma\xi}{\xi-\rho}}\right)(1-\lambda^{\frac{\xi\sigma}{\xi-\rho}})}{\lambda^{\frac{\xi\sigma}{\xi-\rho}}\left(1-a\lambda^{\frac{-\sigma\xi}{\xi-\rho}}\right)}\right]L.$$

### 7.3 Appendix 3

In Appendix 3, we prove that there exists a series of constant numbers, $\vartheta_0, \vartheta_1, \cdots, \vartheta_n, \vartheta_{n+1}, \cdots$; if $0 < K(0) \le \vartheta_0$, the economy will start from producing good 0 only until the capital stock reaches $\vartheta_0$; if $\vartheta_0 < K(0) \le \vartheta_1$, the economy will start from producing goods 0 and 1; if $\vartheta_n < K(0) \le \vartheta_{n+1}$, the economy will start from producing goods $n$ and $n+1$. Furthermore, $K(t_n) = \vartheta_n$ whenever $K(0) < \vartheta_n$.

Now let us characterize the solution to the above dynamic problem when $K_0 \in (0, \vartheta_0]$ while keeping all the other assumptions unchanged. The economy must start by producing good 0 only. The discounted-value Hamiltonian with the Lagrangian multipliers is the following

$$H_0 = \frac{C(t)^{1-\sigma}-1}{1-\sigma}e^{-\rho t} + \eta_0\xi K(t) + \zeta_0^0(L - C(t)).$$

First order condition and K-T condition are

$$\begin{aligned} C(t)^{-\sigma}e^{-\rho t} &= \zeta_0^0; \\ \zeta_0^0(L - C(t)) &= 0; \\ L - C(t) &= 0 \text{ when } \zeta_0^0 > 0. \end{aligned}$$

and

$$\eta_0 = -\frac{\partial H_0}{\partial K} = -\eta_0\xi.$$

They immediately imply that $C^*(t) = L$, which implies that only good 0 is produced. Since no capital is used for production, we have $\dot{K}(t) = \xi K(t)$. When capital stock $K$ exceeds $\vartheta_0$ by an infinitessimal amount, the economy produces both good 0 and good 1. From that point on, the problem is exactly the same as the one we have just solved in the main text. The definition of $t_0$ implies that it is the time point when $K$ just equals $\vartheta_0$. Thus $K_0 e^{\xi t_0} = \vartheta_0$, so $t_0 = \frac{\log \frac{\vartheta_0}{K_0}}{\xi}$. Therefore

$$C^*(t) = \begin{cases} L, & \text{when} \quad t \leq t_0 \\ Le^{\frac{\xi-\rho}{\sigma}(t-t_0)}, & \text{when} \quad t > t_0 \end{cases}.$$

Observe that $Le^{\frac{\xi-\rho}{\sigma}(t_j-t_0)} = C(t_j) = \lambda^j L$, so $t_j = t_0 + \left(\frac{\sigma \log \lambda}{\xi-\rho}\right) j$.

Correspondingly, the capital stock on the equilibrium path is given by

$$K(t) = \begin{cases} K_0 e^{\xi t}, & \text{for} \quad t \in [0, t_0] \\ \frac{-\frac{aL}{\lambda-1}}{\frac{\xi-\rho}{\sigma}-\xi} e^{\frac{\xi-\rho}{\sigma}(t-t_0)} + \frac{-aL}{\xi(\lambda-1)} + \left[\vartheta_0 + \frac{\frac{aL}{\lambda-1}}{\frac{\xi-\rho}{\sigma}-\xi} + \frac{aL}{\xi(\lambda-1)}\right] e^{\xi(t-t_0)}, & \text{for} \quad t \in [t_0, t_1] \\ F(t), & \text{for} \quad t \in [t_n, t_{n+1}], \text{ any } n \geq 1 \end{cases},$$

where

$$F(t) \equiv -\frac{a^{n+1} - a^n}{\lambda^{n+1} - \lambda^n} \left[ \frac{Le^{\frac{\xi-\rho}{\sigma} t}}{\left(\frac{\xi-\rho}{\sigma} - \xi\right)} + \frac{\lambda^n (a-\lambda) L}{\xi (a-1)} \right]$$
$$+ [\lambda^n]^{\frac{-\xi\sigma}{\xi-\rho}} \left\{ K(t_n) + \frac{a^{n+1} - a^n}{\lambda - 1} L \left[ \frac{1}{\left(\frac{\xi-\rho}{\sigma} - \xi\right)} + \frac{(a-\lambda)}{\xi (a-1)} \right] \right\} e^{\xi(t-t_0)},$$

$t_0 = \frac{\log \frac{\vartheta_0}{K_0}}{\xi}$, $t_j = t_0 + \left(\frac{\sigma \log \lambda}{\xi-\rho}\right) j$ for any $j \geq 1$, $K(t_0) = \vartheta_0$, and $K(t_n)$ is exactly the same as before for any $n \geq 1$.

Using the similar algorithm, we can fully specify the transitional dynamics when $K_0 > \vartheta_1$. We have already provided an algorithm to compute $\vartheta_i$ for $i \geq 2$ in the main text by using (41). An alternative way is to back out the threshold value $\vartheta_i$ from the corresponding transversality conditions for any $i \geq 2$. It can be verified that all these values are the same for both algorithms, and that all these threshold values are independent of $K_0$. In other words, $K_0$ only has level effect (i.e. it only affects $C(0)$) but it has no speed effect on industrial upgrading. The main reason is that this economy is perfectly stationary, thus different initial capital levels only translate into different initial aggregate consumption and initial industrial structures.

### 7.4 Appendix 4

#### 7.4.1 Proof of Lemma 1

Observe that when $\theta > 0$,

$$\begin{aligned}\sum_{n=1}^{\infty}\lambda^{\frac{n-1}{1-\gamma}}\left(1+a^{n}\theta\right)^{\frac{1}{\gamma-1}} &< \sum_{n=1}^{\infty}a^{n}\lambda^{\frac{n-1}{1-\gamma}}\left(1+a^{n}\theta\right)^{\frac{1}{\gamma-1}} < \sum_{n=1}^{\infty}a^{n}\lambda^{\frac{n-1}{1-\gamma}}\left(a^{n}\theta\right)^{\frac{1}{\gamma-1}} \\ &= \lambda^{\frac{-1}{1-\gamma}}\theta^{\frac{1}{\gamma-1}}\sum_{n=1}^{\infty}\left(\frac{\lambda}{a^{\gamma}}\right)^{\frac{n}{1-\gamma}},\end{aligned}$$

where the first inequality holds because $a > 1$ and the second inequality holds because $\gamma < 1$. Hence both the denominator and the numerator of the right hand side of (33) must be finite. To prove the existence and uniqueness of $\theta$, first we show that the right hand side of equation (33) is a decreasing function of $\theta$. Note that

$$\frac{\sum_{n=1}^{\infty}a^{n}\lambda^{\frac{n-1}{1-\gamma}}\left(1+a^{n}\theta\right)^{\frac{1}{\gamma-1}}}{\left(\frac{1}{\lambda}\right)^{\frac{1}{1-\gamma}}+\sum_{n=1}^{\infty}\lambda^{\frac{n-1}{1-\gamma}}\left(1+a^{n}\theta\right)^{\frac{1}{\gamma-1}}} = \frac{\sum_{n=1}^{\infty}a^{n}\lambda^{\frac{n-1}{1-\gamma}}\left(1+a^{n}\theta\right)^{\frac{1}{\gamma-1}}}{\sum_{n=1}^{\infty}\lambda^{\frac{n-1}{1-\gamma}}\left(1+a^{n}\theta\right)^{\frac{1}{\gamma-1}}}\frac{\sum_{n=1}^{\infty}\lambda^{\frac{n-1}{1-\gamma}}\left(1+a^{n}\theta\right)^{\frac{1}{\gamma-1}}}{\left(\frac{1}{\lambda}\right)^{\frac{1}{1-\gamma}}+\sum_{n=1}^{\infty}\lambda^{\frac{n-1}{1-\gamma}}\left(1+a^{n}\theta\right)^{\frac{1}{\gamma-1}}},$$

and both terms on the right hand side are positive and decreasing with $\theta$. The proof for why the first term on the right-hand side decreases with $\theta$ is the following. Define

$$\Upsilon_{N}(\theta) \equiv \frac{\sum_{n=1}^{N}a^{n}\lambda^{\frac{n-1}{1-\gamma}}\left(1+a^{n}\theta\right)^{\frac{1}{\gamma-1}}}{\sum_{n=1}^{N}\lambda^{\frac{n-1}{1-\gamma}}\left(1+a^{n}\theta\right)^{\frac{1}{\gamma-1}}}.$$

First it is easy to show that $\Upsilon_{2}(\theta) \equiv \frac{\sum_{n=1}^{2}a^{n}\lambda^{\frac{n-1}{1-\gamma}}(1+a^{n}\theta)^{\frac{1}{\gamma-1}}}{\sum_{n=1}^{2}\lambda^{\frac{n-1}{1-\gamma}}(1+a^{n}\theta)^{\frac{1}{\gamma-1}}} = a+\frac{(a^{2}-a)\lambda^{\frac{2-1}{1-\gamma}}\left(1+a^{2}\theta\right)^{\frac{1}{\gamma-1}}}{\sum_{n=1}^{2}\lambda^{\frac{n-1}{1-\gamma}}(1+a^{n}\theta)^{\frac{1}{\gamma-1}}}$, which obviously decreases with $\theta$. Now for any $N \geq 2$, we can rewrite $\Upsilon_{N}(\theta)$ as

$$\Upsilon_{N}(\theta) = a+\sum_{j=2}^{N}(a^{j}-a^{j-1})\left[\frac{\sum_{n=j}^{N}\lambda^{\frac{n-1}{1-\gamma}}\left(1+a^{n}\theta\right)^{\frac{1}{\gamma-1}}}{\sum_{n=1}^{N}\lambda^{\frac{n-1}{1-\gamma}}\left(1+a^{n}\theta\right)^{\frac{1}{\gamma-1}}}\right].$$

Observe that

$$\frac{\lambda^{\frac{n-1}{1-\gamma}}\left(1+a^{n}\theta\right)^{\frac{1}{\gamma-1}}}{\lambda^{\frac{n}{1-\gamma}}\left(1+a^{n+1}\theta\right)^{\frac{1}{\gamma-1}}} \text{ strictly increases with } \theta \text{ for any } n \geq 1,$$

which immediately implies

$$\left[\frac{\sum_{n=j}^{N}\lambda^{\frac{n-1}{1-\gamma}}\left(1+a^{n}\theta\right)^{\frac{1}{\gamma-1}}}{\sum_{n=1}^{N}\lambda^{\frac{n-1}{1-\gamma}}\left(1+a^{n}\theta\right)^{\frac{1}{\gamma-1}}}\right] \text{ must be strictly decreasing in } \theta \text{ for any } j=2,...,N.$$

So we establish that $\Upsilon_N(\theta)$ strictly decreases with $\theta$. Recall that we have assumed $\lambda < a^{\gamma}$ so $\Upsilon_{\infty}(\theta)$ is positive and finite for any $\theta \in (0, \infty)$. Therefore, $\Upsilon_{\infty}(\theta)$ is strictly decreasing in $\theta$.

We have now proved that the right hand side of equation (33) is a decreasing function of $\theta$. In addition, when $\theta \to 0$, the right hand side of (33) goes to infinity; when $\theta \to \infty$, the right hand side of (33) goes to zero, therefore, (33) can uniquely determine the value of $\theta$ by the mean value theorem. Once $\theta$ is known, $c_1$ can be uniquely determined from (32), while $c_n$ for any other $n \geq 0$ can be uniquely determined by (29) and (30).

#### 7.4.2 Proof of Lemma 2

Note that $\frac{d\ln\frac{c_{n+1}}{c_n}}{dE} = \frac{d\ln\frac{c_{n+1}}{c_n}}{d\theta}\frac{d\theta}{dE}$ and $\frac{d\theta}{dE} < 0$ from Lemma 1. Now, (28) and (30) yield

$$\frac{d\ln\frac{c_{n+1}}{c_n}}{d\theta} = \frac{1}{\gamma-1}\left[\frac{a^{n}\left(a-1\right)}{\left(1+a^{n+1}\theta\right)\left(1+a^{n}\theta\right)}\right] < 0, \forall n \geq 1; \text{and } \frac{d\ln\frac{c_1}{c_0}}{d\theta} < 0,$$

which implies that $\frac{d\ln\frac{c_{n+1}}{c_n}}{dE} > 0$.

#### 7.4.3 Proof of Lemma 3

The factor market clearing conditions now become

**Proof.**

$$\sum_{n=0}^{\infty} c_n = L \tag{46}$$

$$\sum_{n=0}^{\infty} a^{n} c_n = E \tag{47}$$

By contradiction, suppose $\frac{\partial c_0}{\partial E} \geq 0$. Lemma 2 implies that $\frac{\partial c_n}{\partial E} > 0$ for all $n \geq 1$. Differentiating equation (46) with respect to $E$, the derivative of the left hand side is positive but the derivative of the right hand side is zero. That is a contradiction. Thus $\frac{\partial c_0}{\partial E} < 0$. Now we show there must exist some $N$ so that $\frac{\partial c_N}{\partial E} \geq 0$. If it were not true, then we have $\frac{\partial c_n}{\partial E} < 0$ for all $n$. Now differentiating equation (47) with respect to $E$, we would reach a contradiction. Now by revoking Lemma 2, we can conclude that, for any given $E$, there must exist a unique finite positive integer $n^*(E)$ such that $\frac{\partial c_n}{\partial E} \geq 0$ holds if and only if $n > n^*(E)$. ■

Furthermore, we want to show that any industry $n$ will eventually decline as $E$ becomes sufficiently large. First, (29) and (31) imply that for any $n \geq 1$, $c_n'(\theta) > 0$ if and only if $G_n'(\theta) > 0$, where

$$G_n(\theta) \equiv \frac{\left[1 + \sum_{h=1}^{\infty} \lambda^{\frac{h}{1-\gamma}} \left(1 + a^h \theta\right)^{\frac{1}{\gamma-1}}\right]^{\gamma-1}}{1 + a^n \theta}.$$

which is equivalent to

$$\Phi(\theta) \equiv \frac{\sum_{h=1}^{\infty} \lambda^{\frac{h}{1-\gamma}} a^h \left(1 + a^h \theta\right)^{\frac{1}{\gamma-1}-1}}{1 + \sum_{h=1}^{\infty} \lambda^{\frac{h}{1-\gamma}} \left(1 + a^h \theta\right)^{\frac{1}{\gamma-1}}} > \frac{a^n}{1 + a^n \theta} \tag{48}$$

Note that as $\theta$ becomes sufficiently small, the left hand side of inequality (48) goes to infinity. Thus, as $E$ becomes sufficiently large (so $\theta$ becomes sufficiently small), we must have $c_n'(\theta) > 0$, which implies that $c_n'(E) < 0$.

It remains to show that $n^*(E)$ weakly increases in $E$. Alternatively, we want to show that if $c_n'(\theta') > 0$ for some $\theta' > 0$, then $c_n'(\theta) > 0$ holds for any $\theta < \theta'$. Such a property ensures that $c_n(\theta)$ is single peaked in $\theta$. From (29) and (30), we obtain

$$\begin{aligned} c_n &= c_0 \lambda^{\frac{n}{1-\gamma}} \left(1 + a^n \theta\right)^{\frac{1}{\gamma-1}} \Leftrightarrow \\ \ln c_n &= \ln c_0 + \frac{n}{1-\gamma} \ln \lambda + \frac{1}{\gamma - 1} \ln\left(1 + a^n \theta\right) \\ &\Leftrightarrow \frac{c_n'(\theta)}{c_n} = \frac{c_0'(\theta)}{c_0} + \frac{1}{\gamma - 1} \frac{a^n}{(1 + a^n \theta)}. \end{aligned}$$

For any given $\theta$, $c_n'(\theta) > 0$ if and only if

$$\frac{1}{a^n} > \frac{c_0}{c_0'(\theta)\left(1 - \gamma\right)} - \theta.$$

A sufficient condition for the single-peaked property to hold is that the derivative of the right hand side of the above inequality is non-negative, which is ensured if

$$\frac{c_0(\theta)c_0''(\theta)}{(c_0'(\theta))^2} \leq \gamma \tag{49}$$

The inequality (49) is called the *single peaked condition*, which we assume to hold.

**Table A1: Descriptive Statistics for Main Variables**

| | Employment | | | | K/L | | | | TFP | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | mean | max | min | var | mean | max | min | var | mean | max | min | var |
| 1958 | 32.41 | 506.10 | 0.60 | 2237.51 | 40.83 | 522.51 | 0.53 | 2478.06 | 0.83 | 2.78 | 0.01 | 0.09 |
| 1959 | 33.64 | 502.20 | 0.60 | 2289.05 | 40.26 | 527.34 | 0.73 | 2452.88 | 0.85 | 2.72 | 0.02 | 0.09 |
| 1960 | 33.83 | 544.20 | 0.60 | 2296.32 | 41.30 | 533.15 | 0.79 | 2460.62 | 0.84 | 2.53 | 0.01 | 0.09 |
| 1961 | 33.02 | 498.30 | 0.60 | 2149.71 | 43.76 | 542.03 | 0.82 | 2669.53 | 0.84 | 2.34 | 0.01 | 0.08 |
| 1962 | 33.87 | 497.20 | 0.50 | 2209.94 | 44.05 | 560.17 | 0.89 | 2731.52 | 0.86 | 2.15 | 0.01 | 0.08 |
| 1963 | 33.64 | 495.80 | 0.60 | 2075.64 | 44.88 | 555.52 | 0.87 | 2768.25 | 0.88 | 2.39 | 0.02 | 0.08 |
| 1964 | 34.25 | 527.80 | 0.70 | 2136.37 | 45.57 | 566.21 | 0.86 | 2823.20 | 0.89 | 2.33 | 0.02 | 0.08 |
| 1965 | 35.92 | 559.90 | 0.70 | 2302.21 | 46.24 | 574.63 | 0.91 | 3001.59 | 0.90 | 2.36 | 0.02 | 0.08 |
| 1966 | 37.96 | 554.10 | 0.70 | 2488.57 | 46.90 | 609.33 | 0.96 | 3199.78 | 0.90 | 2.11 | 0.02 | 0.08 |
| 1967 | 38.41 | 528.60 | 0.70 | 2503.62 | 48.94 | 647.69 | 1.03 | 3328.21 | 0.91 | 2.06 | 0.02 | 0.07 |
| 1968 | 38.79 | 528.60 | 0.70 | 2578.30 | 50.93 | 672.95 | 1.12 | 3538.65 | 0.91 | 2.09 | 0.02 | 0.07 |
| 1969 | 39.73 | 533.30 | 0.80 | 2564.71 | 51.56 | 676.38 | 1.21 | 3602.63 | 0.91 | 2.03 | 0.02 | 0.07 |
| 1970 | 37.92 | 521.90 | 0.80 | 2252.72 | 54.98 | 706.90 | 1.29 | 3838.25 | 0.89 | 2.05 | 0.02 | 0.06 |
| 1971 | 36.09 | 478.20 | 1.00 | 1965.18 | 58.70 | 746.92 | 1.41 | 4230.33 | 0.90 | 2.00 | 0.02 | 0.06 |
| 1972 | 37.09 | 465.20 | 1.70 | 1957.17 | 59.90 | 758.83 | 1.43 | 4909.20 | 0.93 | 1.98 | 0.02 | 0.06 |
| 1973 | 38.85 | 498.30 | 1.50 | 2164.21 | 60.05 | 762.15 | 1.49 | 5063.86 | 0.95 | 1.94 | 0.02 | 0.06 |
| 1974 | 38.58 | 514.00 | 1.60 | 2150.54 | 61.70 | 818.13 | 1.67 | 5181.77 | 0.93 | 1.90 | 0.03 | 0.06 |
| 1975 | 35.31 | 447.90 | 1.50 | 1807.61 | 68.37 | 854.93 | 1.83 | 5749.33 | 0.90 | 1.92 | 0.03 | 0.05 |
| 1976 | 36.36 | 448.60 | 1.60 | 1893.39 | 69.01 | 889.04 | 1.80 | 6201.87 | 0.92 | 1.88 | 0.04 | 0.05 |
| 1977 | 38.08 | 438.60 | 1.60 | 2011.51 | 69.41 | 868.07 | 1.87 | 6628.49 | 0.93 | 1.81 | 0.05 | 0.05 |
| 1978 | 39.57 | 440.40 | 1.70 | 2130.00 | 69.65 | 909.67 | 1.91 | 7056.39 | 0.93 | 1.87 | 0.06 | 0.04 |
| 1979 | 40.60 | 447.90 | 1.50 | 2282.34 | 71.45 | 958.42 | 1.91 | 7912.64 | 0.93 | 1.76 | 0.08 | 0.04 |
| 1980 | 39.59 | 400.20 | 1.60 | 2182.67 | 75.34 | 962.61 | 1.88 | 8290.25 | 0.92 | 1.72 | 0.08 | 0.04 |
| 1981 | 38.82 | 387.90 | 1.60 | 2141.37 | 78.41 | 1043.47 | 1.99 | 8858.32 | 0.92 | 1.64 | 0.08 | 0.03 |
| 1982 | 36.48 | 319.90 | 1.70 | 1910.36 | 85.18 | 1111.60 | 2.17 | 10232.88 | 0.91 | 1.69 | 0.09 | 0.03 |
| 1983 | 35.67 | 328.40 | 1.60 | 1836.98 | 88.39 | 1144.69 | 2.15 | 11348.16 | 0.92 | 1.78 | 0.09 | 0.03 |
| 1984 | 36.55 | 345.10 | 1.60 | 1954.74 | 88.47 | 1195.06 | 2.36 | 12046.55 | 0.94 | 2.25 | 0.10 | 0.03 |
| 1985 | 35.74 | 371.50 | 1.60 | 1934.96 | 92.52 | 1239.11 | 2.59 | 13132.13 | 0.94 | 1.92 | 0.09 | 0.03 |
| 1986 | 34.80 | 379.70 | 1.50 | 1896.44 | 97.39 | 1315.15 | 2.74 | 15350.32 | 0.93 | 1.53 | 0.09 | 0.02 |
| 1987 | 35.97 | 384.60 | 1.40 | 2079.68 | 97.47 | 1226.61 | 2.56 | 15551.95 | 0.96 | 1.58 | 0.10 | 0.02 |
| 1988 | 36.41 | 394.20 | 1.40 | 2091.20 | 95.93 | 1222.48 | 2.61 | 14265.56 | 0.97 | 1.56 | 0.11 | 0.02 |
| 1989 | 36.18 | 402.60 | 1.50 | 2093.26 | 96.50 | 1237.15 | 2.74 | 13983.01 | 0.96 | 1.63 | 0.12 | 0.02 |
| 1990 | 35.67 | 403.90 | 1.40 | 2050.80 | 98.62 | 1238.72 | 2.79 | 13572.43 | 0.95 | 1.60 | 0.13 | 0.02 |
| 1991 | 34.05 | 388.50 | 1.40 | 1833.41 | 102.76 | 1272.94 | 2.87 | 13849.51 | 0.93 | 1.62 | 0.15 | 0.02 |
| 1992 | 34.38 | 425.60 | 1.30 | 1920.59 | 103.22 | 1337.73 | 2.89 | 14691.81 | 0.96 | 1.61 | 0.18 | 0.02 |
| 1993 | 34.33 | 443.00 | 1.30 | 1893.55 | 104.90 | 1374.95 | 2.76 | 15665.73 | 0.96 | 1.58 | 0.23 | 0.01 |
| 1994 | 34.55 | 472.10 | 1.20 | 1955.89 | 107.00 | 1471.50 | 2.84 | 17114.33 | 0.98 | 1.61 | 0.32 | 0.01 |
| 1995 | 35.30 | 499.80 | 1.20 | 2071.01 | 107.36 | 1431.76 | 2.81 | 17391.41 | 0.99 | 1.59 | 0.57 | 0.01 |
| 1996 | 35.13 | 499.60 | 1.20 | 2068.94 | 109.29 | 1361.83 | 3.03 | 17538.76 | 0.98 | 1.90 | 0.61 | 0.01 |
| 1997 | 35.53 | 522.70 | 1.30 | 2136.95 | 113.23 | 1574.08 | 3.58 | 22057.96 | 1.00 | 1.00 | 1.00 | 0.00 |
| 1998 | 35.83 | 534.70 | 1.20 | 2209.45 | 120.77 | 1577.91 | 5.33 | 24788.13 | 0.98 | 1.28 | 0.69 | 0.00 |
| 1999 | 35.28 | 550.90 | 0.90 | 2204.82 | 132.30 | 1369.66 | 8.32 | 27692.76 | 0.96 | 1.81 | 0.72 | 0.01 |
| 2000 | 35.21 | 554.90 | 0.90 | 2259.85 | 142.19 | 1621.59 | 10.91 | 33308.42 | 0.95 | 2.10 | 0.63 | 0.01 |
| 2001 | 33.50 | 528.60 | 0.90 | 2099.48 | 154.64 | 1975.08 | 14.23 | 38101.63 | 0.91 | 2.39 | 0.47 | 0.02 |
| 2002 | 31.05 | 488.30 | 0.90 | 1800.05 | 169.16 | 1598.52 | 10.11 | 39074.01 | 0.93 | 3.29 | 0.46 | 0.04 |
| 2003 | 29.33 | 478.00 | 0.80 | 1626.39 | 180.39 | 2289.47 | 11.72 | 47223.99 | 0.93 | 3.80 | 0.49 | 0.05 |
| 2004 | 28.32 | 448.50 | 0.80 | 1546.33 | 190.17 | 2333.50 | 12.61 | 50311.07 | 0.95 | 4.57 | 0.43 | 0.06 |
| 2005 | 27.84 | 446.00 | 0.80 | 1510.21 | 193.72 | 2224.50 | 15.06 | 50917.25 | 0.97 | 6.29 | 0.42 | 0.10 |
| All | 35.52 | 559.90 | 0.50 | 2080.82 | 88.09 | 2333.50 | 0.53 | 15433.13 | 0.93 | 6.29 | 0.01 | 0.05 |

Note: All variables are from NBER-CES Manufacturing Industry Data for the U.S., which covers 473 industries. Employment is in the unit of 1000 employees. K/L is the ratio of Real Capital Stock to Employment and in the unit of thousand US dollars per worker.