Public Goods Provision: Unit-by-Unit Cost-sharing Rules and the Core[1]

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Abstract

In a simple model of one public good producible from one input, we show that *unit-by-unit cost-sharing rules* have two properties. First, a *unit-by-unit cost-sharing rule* always chooses a core allocation. Second, every allocation in the core will be chosen by at least one *unit-by-unit cost-shring rule*. We argue that costs should be shared on a unit-by-unit basis. We propose a simple mechanism to implement the family of unit-by-unit rules.
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1 Introduction

In this paper, we consider a simple model of producing one public good from one input. The core for this model is non-empty. Some solution concepts or cost-sharing rules always choose allocations in the core. For example, the *Public-Good-Equivalent solution* and the *Ratio Equilibrium* always lie in the core[1]. On the other hand, some solutions occasionally choose allocations outside of the core. The *Proportional cost sharing rule*, which allocates costs proportional to individual benefits, may fail to choose an allocation within the core for some utility profiles.

This paper defines a family of cost-sharing rules with two properties. First, each cost-sharing rule in this family is single valued and always chooses a core allocation. Second, given any utility profile, every allocation in the core is chosen by at least one cost-sharing rule in this family. We call these unit-by-unit rules. In a unit-by-unit rule, agents first share the cost of producing the first unit of the public good based on their willingness to pay for the first unit; the agents then share the cost of producing the second unit based on their willingness to pay for the second unit; and so on. Each agent's total cost-share is the sum of his cost-share for each unit up to the socially efficient production level. We show that an allocation is in the core if and only if it is chosen by at least one unit-by-unit rule. When an allocation is not in the core, it is objected by some coalition of agents, which means that this coalition of agents can do better by providing the public good alone (sharing the cost among this coalition) and letting others consume the public good for free. Allocations outside the core typically are difficult to implement. This result suggests that for the provision of public goods, the cost should be distributed on a unit-by-unit basis.

Examples of public goods available in multiple units are abundant. Consider roommates' joint decision about subscribing to cable TV service. Whether to subscribe to the basic service, the standard service, or the standard service with premium channels and how to share the cost is an example of the provision of a public good available in multiple units. Other examples include community security services at different security levels, a community play ground or swimming pool of different sizes, etc.

We propose a simple mechanism to implement the family of unit-by-unit rules under

[1] See Moulin [1986] for definitions for Public-Good-Equivalent solution and the Ratio Equilibrium.

some restrictions. We focus on implementing deterministic cost-sharing rules, i.e., rules that each picks a single-valued selection of the core. Along this line, for a binary public good, Jackson and Moulin [1992] propose a simple two-stage game that implements a family of cost-sharing rules that always picks a core allocation. Bag and Winter [1999] propose a simple subscription mechanism for an excludable public good that implements the Shapley value or any boundary allocation of the core depending on the ordering of the agents. We are interested in implementing the entire core individually.

In the literature of implementing the core, Bagnoli and Lipman [1989] propose a simple voluntary-contribution mechanism to implement the core in undominated perfect equilibrium. Perry and Reny [1994] propose a coalition formation game to implement the core in stationary subgame perfect equilibrium. Moldovanu [1996] proposes a similar coalition formation game to implement the core if the public good is excludable in subgame perfect Nash equilibrium. Implementing the entire core means that any allocation in the core is an equilibrium of the proposed game. In comparison with the literature on implementing the entire core, the mechanism proposed in this paper only implements one allocation in the core and every allocation in the core can be implemented by adjusting this mechanism accordingly.

2 Unit-by-Unit Cost-sharing Rules

We study a model with one public good producible from a single input: money. This public good is discrete[2]. For ease of disposition, we further assume that g can only take the value of integers. Let $c(g)$ denote the cost of producing the gth unit of the public good. There is no fixed cost, $c(0) = 0$, and $c(g)$ is increasing in g. Let $C(g)$ denote the total cost of producing g units of the public good. Every production technology has finite capacity. Let $\overline{\overline{g}}$ denote the largest capacity for all possible production technologies. Define domain $\mathcal{C}$ as follows:

$$\mathcal{C} = \{c : c(0) = 0, c(g') > c(g) \text{ if } g' > g \text{ and } C(g) = \infty \text{ if } g > \overline{\overline{g}}\}.$$

There are n consumers with quasi-linear utilities. Let N be the set of consumers. Let

[2] We present only the results for discrete public goods for simplicity and because we gain no additional insight from analyzing the divisible-good model. Results can be generalized to the divisible-good model and are available in an earlier version of this paper.

$U_i(g, x_i) = U_i(g) - x_i$ denote agent i's utility from consuming g units of the public good and paying x_i. Let u_i denote agent i's marginal utility function. Let $u_i(g)$ denote agent i's willingness to pay for the gth unit of the public good. $u_i(g)$ is non-increasing in g. Throughout this paper, $C(g)$ denotes the total cost, $c(g)$ denotes the marginal cost; while $U_i(g)$ denotes the total utility and $u_i(g)$ denotes agent i's marginal utility. Let g^* denote the optimal production level: $g^* = \max\{g : \sum_i u_i(g) - c(g) > 0\}$.

Define domain B as follows:

$$B = \{(u_1, ..., u_n) : \forall i,\ u_i \text{ is non-increasing in } g \text{ and } g^* > 0.\}$$

Throughout this paper, we study only cost-sharing rules that are budget-balanced, efficient, voluntary, and with non-negative cost shares. We therefore build those properties into the definition of cost-sharing rules.

Definition 1: Given $c \in C$, a **cost-sharing rule** is a mapping, γ, associating to every utility profile, $(u_1, .., u_n) \in B$, a vector of cost shares, $\gamma_i(u_1, ..., u_n)$, $i = 1, ..., n$, and satisfying:

Budget Balance: $\sum_{i=1}^{n} \gamma_i(u_1, ..., u_n) = C(g^*)$, and

No Subsidy: $0 \leq \gamma_i(u_1, ..., u_n) \leq U_i(g^*)$ for all i;

where $g^* = \max\{g : \sum_i u_i(g) - c(g) > 0\}$.

To define *unit-by-unit rules*, we first define the *one-unit cost-sharing rule*.

Definition 2: Let w_i be agent i's willingness to pay for one unit of a public good and let c^1 be the cost of producing that unit of the public good. Define domain W such that it is efficient to provide that unit of the public good, i.e., $W = \{(w_1, ..., w_n) : w_i \geq 0\, \forall i,\ \sum_i w_i > c^1\}$. A **one-unit cost-sharing rule** is a mapping, s, associating to every willingness to pay profile, $(w_1, ..., w_n) \in W$, a vector of cost shares $s_i(w_1, ..., w_n; c^1)$, $i = 1, ..., n$, and satisfying:

Budget Balance: $\sum_i s_i(w_1, ..., w_n; c^1) = c^1$, and

No Subsidy: $0 \leq s_i(w_1, ..., w_n; c^1) \leq w_i$ for all i,

Cross Monotonicity: for any $w_j < w'_j$, $s_i(w_1, ..., w_j, ..., w_n; c^1) \leq s_i(w_1, ..., w'_j, ..., w_n; c^1)$ for any $i \neq j$.

Cross Monotonicity requires that an agent's cost share doesn't increase when another agent's willingness to pay increases. Cross Monotonicity plays an important role in our implementation result, but it is not necessary for the characterization result (Theorem 1 and

2). Let S be the set of all one-unit cost-sharing rules defined above. A *unit-by-unit rule* prefixes a one-unit cost-sharing rule for each unit. The prespecified one-unit cost-sharing rules for different units could be different. For any admissible utility profile, a unit-by-unit rule determines the cost-share for each agent in two steps. First, it calculates the optimal production level. Then, for each agent, it assigns a cost-share that equals the sum of this agent's cost-shares according to those prespecified one-unit cost-sharing rules for all the units up to the optimal production level.

Definition 3: Given $c \in \mathcal{C}$, a cost-sharing rule is in the **family of unit-by-unit rules** if it is a mapping, f, associating to every utility profile, $(u_1, ..., u_n) \in B$, a vector of cost shares $f_i(u_1, ..., u_n)$, $i = 1, ..., n$, and satisfying:

Unit-by-Unit: There exists a vector of one-unit cost sharing rules $(s^1, ..., s^{\overline{\overline{g}}})$ where $s^g \in S$ for all g, such that for every utility profile, $(u_1, ..., u_n) \in B$, we have:

$$f_i(u_1, ..., u_n) = \sum_{g=1}^{g^*} s_i^g(u_1(g), ..., u_n(g); c(g)) \text{ for all } i, \text{ where } g^* = \max\{g : \sum_i u_i(g) - c(g) > 0\}.$$

Unit-by-unit rules allocate costs on a unit-by-unit basis. It is clear that a unit-by-unit rule is a cost-sharing rule: it satisfies both *budget balance* and *no subsidy.* In a unit-by-unit rule, for the gth unit, s^g is fixed for all utility profiles in B. For different g, the prespecified s^g need not be the same. If s^g is the proportional cost-sharing rule for every g, i.e., $s_i^g = \frac{u_i(g)}{\sum_j u_j(g)}$, the corresponding unit-by-unit rule is called a *unit-by-unit proportional cost-sharing rule.* For every unit of the public good up to the efficient level, this rule allocates the cost proportional to each agent's willingness to pay for that unit. Yu (2004) characterizes the unit-by-unit proportional cost-sharing rule. The unit-by-unit proportional cost-sharing rule always chooses an allocation in the core, as does any other unit-by-unit rule.

Definition 4: Given a public good provision problem, $(u_1, ..., u_n; c)$, let $(g; x_1, ...x_n)$ be a feasible allocation: i.e., $x_i \geq 0$ and $\sum_i x_i \geq C(g)$. The allocation $(g; x_1, ...x_n)$ is in the **core** if there exists no coalition, $T \subseteq N$, such that there is an allocation $(g'; (x'_j, j \in T))$ that satisfies: (1) feasibility: $\sum_T x'_j \geq C(g')$, and $x'_j \geq 0$, for all $j \in T$; (2) Pareto improvement: $U_j(g') - x'_j \geq U_j(g) - x_j$ for all $j \in T$ with at least one strict inequality.

Theorem 1: For any $c \in \mathcal{C}$ and any $u = (u_1, ..., u_n) \in B$, if f is a unit-by-unit rule, then the allocation according to f, $(g^*; f_1(u), ..., f_n(u))$, is in the core.

Proof: Let $(g^*; f_1(u), ..., f_n(u)) = (g^*; x_1, ..., x_n)$ be the allocation chosen by a unit-by-unit rule, f. Notice first that g^* is the optimal production level, i.e., $g^* = \max\{g : \sum_i u_i(g) - c(g) > 0\}$. Suppose that $(g^*; x_1, ..., x_n)$ is not in the core. Then, there exists a coalition, $T \subseteq N$, such that there is an allocation $(g'; (x'_j, j \in T))$ that satisfies: (1) $\sum_T x'_j \geq C(g')$ and $x'_j \geq 0$ for all $j \in T$; (2) $U_j(g') - x'_j \geq U_j(g^*) - x_j$ for all $j \in T$, with at least one strict inequality. We distinguish two cases:

(I) $g' \leq g^*$.

Since f is a unit-by-unit rule, we know that there exists a vector of one-unit cost-sharing rules, $(s^1, s^2, ..., s^{g^*}, ...)$, such that

$$f_i(u_1, ..., u_n) = \sum_{g=1}^{g^*} s_i^g(u_1(g), ..., u_n(g); c(g)) \text{ for all } i \in N.$$

Let $y_i = \sum_{g=1}^{g'} s_i^g(u_1(g), ..., u_n(g); c(g))$ for all i. Notice that $\sum_{i \in N} y_i = C(g')$.

Because $s_i^g(u_1(g), ..., u_n(g); c(g)) \leq u_i(g)$ for all $g \in \{g', ..., g^*\}$ and all $i \in N$, we have:

$$U_i(g^*) - x_i \geq U_i(g') - y_i \text{ for all } i \in N.$$

For all $j \in T$, we have $U_j(g') - x'_j \geq U_j(g^*) - x_j \geq U_j(g') - y_j$. Therefore, $x'_j \leq y_j$ for all $j \in T$, with at least one strict inequality. Consequently,

$$C(g') \leq \sum_T x'_j < \sum_T y_j \leq C(g').$$

Contradiction.

(II) $g' > g^*$.

We have: $\sum_T U_j(g') - C(g') \geq \sum_T (U_j(g') - x'_j) > \sum_T (U_j(g^*) - x_j) \geq \sum_T U_j(g^*) - C(g^*)$.

For all $i \in N/T$, we have $U_i(g') \geq U_i(g^*)$. Therefore,

$$\sum_N U_i(g') - C(g') > \sum_N U_i(g^*) - C(g^*).$$

This is contradictory to the definition of g^*.

Therefore, $(g^*; f_1(u), ..., f_n(u))$ is in the core. ∎

Since every allocation chosen by a unit-by-unit rule is in the core, apparently any allocation outside the core cannot be supported (chosen) by any unit-by-unit rule. What about allocations in the core? Can every allocation in the core be chosen by at least one unit-by-unit rule? Theorem 2 addresses this question.

Theorem 2: For any $c \in C$ and any $(u_1, ..., u_n) \in B$, if $(\widehat{g}; x_1, ..., x_n)$ is in the core, then there exists a unit-by-unit rule, f, such that $x_i = f_i(u_1, ..., u_n)$ for all $i \in N$.

Proof:

Now, let $(\widehat{g}; x_1, ..., x_n)$ belong to the core. First, notice that $\widehat{g} = g^* = \max\{g : \sum_i u_i(g) - c(g) > 0\}$ due to the Pareto optimality of all allocations in the core. If $g^* = 1$, let $f_i(u_1, ..., u_n) = s_i^1(u_1(1), ..., u_n(1); c(1)) = x_i$ and Theorem 2 holds. Now, consider the case where $g^* > 1$. Define agent i's *adjusted willingness to pay* $H_i(g)$ such that $U_i(g) - H_i(g) = U_i(g^*) - x_i$. Notice that $H_i(g)$ is non-decreasing in g. By the definition of the core,

$$\sum_{i \in S} U_i(g) - C(g) \leq \sum_{i \in S} (U_i(g^*) - x_i) \text{ for all } g = 1, 2, ..., g^*, ... \text{ and all } S \subseteq N.$$

Therefore:

$$\sum_{i \in S} H_i(g) \leq C(g) \text{ for all } g = 1, 2, ..., g^*, ... \text{ and all } S \subseteq N.$$

Let $H_i^+(g) = \max\{0, H_i(g)\}$. For any g, let $T_g = \{i \in N : H_i(g) > 0\}$. We have:

$$\sum_{i \in S} H_i^+(g) \leq \sum_{i \in N} H_i^+(g) = \sum_{i \in T_g} H_i(g) \leq C(g) \text{ for all } g = 1, 2, ..., g^*, ... \text{ and all } S \subseteq N. \quad (1)$$

Define $\widetilde{H}_i(t)$ as follows:

$$\begin{aligned} \widetilde{H}_i(t) &= H_i^+(t) \text{ for all } t = 1, 2, ..., g^*, ...; \\ \widetilde{H}_i(t) &= H_i^+(k) + (t - k)[H_i^+(k+1) - H_i^+(k)] \text{ for all } t \in (k, k+1), k = 1, 2, ..., g^*, ... \end{aligned}$$

Notice that t is not restricted to integers. The proof strategy is to construct a unit-by-unit rule that chooses $(\widehat{g}; x_1, ..., x_n)$. Intuitively, we can think of a water tank (at a high place) connected with n vertically placed tubes on the ground (one for each agent) such that when the height of the tubes are t, the capacity of agent i's tube is $\widetilde{H}_i(t)$. See Figure 1 for an illustration.

[insert Figure 1 about here]

Notice that each tube has zero capacity until it reaches a certain height, t. This is because $\widetilde{H}_i(t) = 0$ when agent i's adjusted willingness to pay $H_i(t)$ is non-positive. To distribute the cost of the first unit, $c(1)$, we set a valve on every agent's tube such that the amount of water in agent i's tube will not exceed $u_i(1)$ and then pour $c(1)$ amount of water into the water tank. Water flows to the n tubes. The amount of water in agent i's tube is agent i's cost-share for the first unit. To distribute the cost of the second unit, $c(2)$, we adjust the valves such that the additional capacity (additional to the amount of the *existing water* in that tube) of agent i's tube does not exceed $u_i(2)$ and then pour an additional $c(2)$ amount of water into the water tank. Water flows to the n tubes. The amount of *additional* water in agent i's tube is agent i's cost-share for the second unit. We do the same for every unit up to g^*. Every time we pour $c(g)$ amount of water into the water tank, the incremental amount of water in agent i's tube is agent i's cost-share for the gth unit. After g^* units, the total amount of water in agent i's tube should be x_i and the height of the water is g^*. That is true for every agent i. The formal description of this system is as follows:

Because $\sum_{i\in N} \widetilde{H}_i(1) \leq c(1)$ and $\sum_{i\in N} u_i(1) > c(1)$, there exists a $t_1 \geq 1$ such that:

$$\sum_{i\in N} \min\{u_i(1), \widetilde{H}_i(t_1)\} = c(1).$$

Let $y_i(1) = \min\{u_i(1), \widetilde{H}_i(t_1)\}$. By design, every agent should be at the same level, t, of his adjusted willingness to pay, $\widetilde{H}_i(t)$, unless his cost-share has reached his benefit from that unit.

Because $\sum_{i\in N} \widetilde{H}_i(2) \leq C(2)$, $\sum_{i\in N} y_i(1) = c(1)$, and $\sum_{i\in N} u_i(2) > c(2)$, there exists a $t_2 \geq 2$ such that:

$$\sum_{i\in N} \min\{u_i(2) + y_i(1), \widetilde{H}_i(t_2)\} = C(2).$$

Let $y_i(2) = \min\{u_i(2) + y_i(1), \widetilde{H}_i(t_2)\} - y_i(1)$, $\forall i$.

Define $y_i(k)$ for all $k = 3, 4, ..., g^*$, successively. When $y_i(k-1)$ is defined, there exists a $t_k \geq k$ such that:

$$\sum_{i\in N} \min\{u_i(k) + \sum_{g=1}^{k-1} y_i(g), \widetilde{H}_i(t_k)\} = C(k), \tag{2}$$

because $\sum_{i\in N} \widetilde{H}_i(k) \leq C(k), \sum_{g=1}^{g=k-1} \sum_i y_i(g) = C(k-1)$, and $\sum_{i\in N} u_i(k) > c(k)$.

We prove two claims.

Claim 1: $t_1 < t_2 < ... < t_{g^*}$.

The assumptions of increasing marginal cost and non-decreasing marginal utility play important roles in the proof of Claim 1. If Claim 1 holds, we have $y_i(k) \geq 0$ for all $i \in N$ and all $k = 1, 2, ..., g^*$. By the design of the system, we have $y_i(k) \leq u_i(k)$. Hence, $0 \leq y_i(k) \leq u_i(k)$ for all $i \in N$ and all $k = 1, 2, ..., g^*$.

Claim 2: $\sum_{g=1}^{k} y_i(g) \geq \max\{x_i - [U_i(g^*) - U_i(k)], 0\} = H_i^+(k)$ for all $i \in N$ and all $k = 1, 2, ..., g^*$.

A proof of the claims is provided in the appendix.

According to Claim 2, $\sum_{g=1}^{g^*} y_i(g) \geq H_i^+(g^*)$ for all $i \in N$. According to the definition of $H_i^+(g)$, we have $H_i^+(g^*) = x_i$ for all $i \in N$. Thus, $\sum_{g=1}^{g^*} y_i(g) \geq x_i$ for all $i \in N$. We also have: $\sum_{i \in N} \sum_{g=1}^{g^*} y_i(g) = C(g^*) = \sum_{i \in N} x_i$. Therefore, it must be true that $\sum_{g=1}^{g^*} y_i(g) = x_i$ for all $i \in N$.

For every $g \leq g^*$, choose a one-unit cost-sharing rule, s^g, such that $s_i^g(u_1(g), ..., u_n(g); c(g)) = y_i(g)$ for all $i \in N$. Let $f_i(u_1, ..., u_n) = \sum_{g=1}^{g^*} y_i(g)$ for all $i \in N$. f is a unit-by-unit rule that selects the core allocation $(\widehat{g}, x_1, ..., x_n)$ when the utility profile is $(u_1, ..., u_n)$. ■

Combining Theorems 1 and 2 yields a characterization result for the core in this public goods provision model: an allocation is in the core if and only if it is chosen by at least one unit-by-unit rule. In a unit-by-unit rule, the cost of producing every unit of the good is shared according to each agent's willingness to pay for that unit and the cost shares are **independent** of each agent's willingness to pay for other units. If an allocation is not in the core, it is objected by a group of agents and, in general, it is difficult to implement any allocation outside the core. Our characterization result indicates that costs should be distributed on unit-by-unit basis.

To illustrate the concept of unit-by-unit rules, we present a numerical example and discuss the difference between two traditional cost-sharing rules and their unit-by-unit versions.

Example: There are two agents, A and B. There is one public good: the cost of producing the first unit, $c(1)$, is 2; the cost of producing the second unit, $c(2)$, is 10; the cost of producing the third unit, $c(3)$, is infinite. The agents' willingness to pay is the following: $u_A(1) = 20$, $u_A(2) = 10$; $u_B(1) = 20$, $u_B(2) = 2$. Let x_i denote agent i's cost share. The core in this example is to produce 2 units of the public good and to allocate the cost such that: $8 \leq x_A \leq 12$; $0 \leq x_B \leq 4$; and $x_A + x_B = 12$.

A *proportional cost-sharing rule* allocates costs proportional to each agent's total benefit from consumption. In this example, according to the proportional cost-sharing rule, the agents' cost shares are $x_A = \frac{30}{30+22} * 12 = 6.92$, $x_B = \frac{22}{30+22} * 12 = 5.08$. This allocation is not in the core. A *unit-by-unit proportional cost-sharing rule* allocates the cost of each unit proportional to each agent's benefit from that unit. In this example, according to the unit-by-unit proportional cost-sharing rule, $x_A = \frac{20}{20+20} * 2 + \frac{10}{10+2} * 10 = 9.33$, $x_B = \frac{20}{20+20} * 2 + \frac{2}{10+2} * 10 = 2.67$, and it is in the core.

Equal-cost-with-no-exploitation (ECNE) is a cost-sharing rule under which each agent pays either a common amount or his total willingness to pay, whichever is less. In this example, according to ECNE, the agents' cost shares are: $x_A = x_B = 6$. Again, this allocation is not in the core. A *unit-by-unit ECNE* allocates the cost of each unit according to ECNE. In this example, the unit-by-unit ECNE chooses the core allocation: ($x_A = \frac{2}{2} + (10 - 2) = 9$, $x_B = \frac{2}{2} + 2 = 3$).

3 Implementing unit-by-unit cost-sharing rules

We propose a mechanism to implement any unit-by-unit cost-sharing rule when agents' preferences can be characterized by their types that satisfy the sorting condition: marginal utility is monotonic in type. It is the "single crossing" or "Spence-Mirrlees " condition applied to the case of quasilinear utility.

Our mechanism is similar to the mechanism proposed by Jackson and Moulin (1992) for a binary public good. Bag (1997) generalized the Jackson-Moulin mechanism to a divisible public good while agents have constant marginal utility. Yu (2003) generalized it further to the case when each agent's utility takes the form $\lambda_i v(g)$. All the three mentioned papers use implementation in undominated Nash equilibrium. This paper uses implementation in undominated subgame perfect Nash equilibrium.

Let λ_i denote agent i's type. Agent i's utility takes the following form:

$$U(g; \lambda_i) - x_i = \sum_{k=1}^{k=g} v(k; \lambda_i) - x_i,$$

where as before g is the level of the public good and x_i is agent i's cost share. The function v is the same for all agents and is public knowledge. $v(g, \lambda_i)$ is non-increasing and non-negative in g for all λ_i. For all g, $v(g, \lambda_i)$ is continuous in λ_i. Moreover, $v(g, \lambda_i)$ satisfies the sorting condition: it is strictly increasing in λ_i for all g.

Let f be a unit-by-unit cost-sharing rule defined in the previous section. Let $F_i(\lambda_1, ..., \lambda_n) = f_i(u_1, ..., u_n)$ for all $i \in N$, all $\lambda_i \geq 0$, where $u_i(\cdot) = v(\cdot, \lambda_i)$. It follows that F is a unit-by-unit cost-sharing rule.

Let $g^*(\lambda_1, ..., \lambda_n) = \max\{g : \sum_i v(g, \lambda_i) > c(g)\}$ be the optimal production level. Let $b^*(\lambda_1, ...\lambda_n) = \sum_i v(g^*, \lambda_i)$ be the joint benefit at the optimal production level. Now define an auxiliary function, $\widetilde{\lambda}_i$, for all $i \in N$, all $g > 0, b \geq 0$, and $\lambda_j \geq 0$, $j \neq i$. Intuitively, $\widetilde{\lambda}_i(g, b; \lambda_{-i})$ is agent i's inferred type based on other agents' types and the optimal production level:

$\widetilde{\lambda}_i(g, b; \lambda_{-i}) = 0$, if $g^*(\lambda_1, ...\lambda_i' = 0, ...\lambda_n) > g$ or $g^*(\lambda_1, ...\lambda_i' = 0, ...\lambda_n) = g$ and $b^*(\lambda_1, ...\lambda_i' = 0, ...\lambda_n) > b$;

otherwise, $\widetilde{\lambda}_i(g, b; \lambda_{-i}) = \lambda_i'$ such that $g^*(\lambda_1, ...\lambda_i', ...\lambda_n) = g$ and $b^*(\lambda_1, ...\lambda_i', ...\lambda_n) = b$.

We can find such a λ_i' due to the continuity of $v(\cdot, \lambda_i)$ in λ_i.

Now, we are ready to define an auxiliary function θ_i for all $i \in N$,all $g > 0, b \geq 0$, and $\lambda_j \geq 0$, $j \neq i$. Intuitively, $\theta_i(g, b; \lambda_{-i})$ is agent i's cost share based on his inferred type and other agents' types:

$$\theta_i(g, b; \lambda_{-i}) = F_i(\lambda_1, ..., \widetilde{\lambda}_i(g, b; \lambda_{-i}), ..., \lambda_n).$$

The Mechanism:

Stage 1. Each agent bids (simultaneously) the optimal production level and the joint marginal utility at the optimal production level. Denote agent i's bid (g_i, b_i).

The agent with the highest g_i wins. When there is more than one agent with highest g_i, then the agent with the highest b_i and g_i wins. When there is more than one agent with the highest g_i and b_i, a winner is chosen randomly among those with the highest g_i and b_i. Denote the winner i^*, and the winning bid $(\widehat{g}, \widehat{b})$.

If $\widehat{g} \leq 0$, STOP;

If $\widehat{g} > 0$, go to stage 2.

Stage 2. Each agent reports his type (simultaneously). Denote agent i's report β_i.

Denote β_N the vector of all agents' report. Let $V(\widehat{g}; \beta_N) = \sum_{i \in N} v(\widehat{g}; \beta_i)$.

(a) If $V(\widehat{g};\beta_N) - c(\widehat{g}) > \widehat{b}$, $\widehat{g}$-level of the public good is provided. Agent i, $i \neq i^*$, pays $\theta(\widehat{g},\widehat{b};\beta_{-i})$ and consumes $\widehat{g}$-level of the public good. Agent i^* pays the balance (namely, $C(\widehat{g}) - \sum_{i\neq i^*}\theta(\widehat{g},\widehat{b};\beta_{-i})$).

(b) If $V(\widehat{g};\beta_N) - c(\widehat{g}) < \widehat{b}$, there will be no production. Agent i, $i \neq i^*$, receives a transfer

$$t_i = U(\widehat{g};\widetilde{\lambda}_i(\widehat{g},\widehat{b};\beta_{-i})) - \theta(\widehat{g},\widehat{b};\beta_{-i}) \qquad \text{from agent } i^*.$$

(c) If $V(\widehat{g};\beta_N) - c(\widehat{g}) = \widehat{b}$, agent i^* can choose either of the above alternatives.

The equilibrium concept used here is undominated sub-game perfect Nash equilibrium. An equilibrium is an undominated sub-game perfect Nash equilibrium if it is a sub-game perfect Nash equilibrium and no agent uses a dominated strategy in any sub-game.

Note that in case (b), the amount of transfer t_i is specifically designed so that truthful reporting of his type is a dominant strategy for any non-leading agent in Stage 2.

Theorem 3: Let F be any unit-by-unit cost-sharing rule. The above mechanism implements F in undominated subgame perfect Nash equilibrium.

Proof of the Theorem 3

(a) We analyze Stage 2 first

Suppose agent i^* won stage 1 with a bid $(\widehat{g},\widehat{b})$. We call agent i^* the leading agent (or the leader in Stage 2) and all other agents the followers. We first check that reporting truthfully is a dominant strategy for every follower, $i \neq i^*$.

Set $\alpha = \widetilde{\lambda}_i(\widehat{g},\widehat{b};\beta_{-i})$. Note that α is agent i's inferred type based on the winning bid in Stage 1 and other agents' reports in Stage 2. The key point is that if agent i's type report is greater than his inferred type, the public good is produced and agent i's cost share is independent of his own report.

If $\beta_i > \alpha$, then $\widehat{g}$ units of the public good is produced and agent i's utility is $U(\widehat{g};\lambda_i) - \theta(\widehat{g},\widehat{b};\beta_{-i})$.

If $\beta_i < \alpha$, then no public good is produced and agent i's utility is $U(\widehat{g};\alpha) - \theta(\widehat{g},\widehat{b};\beta_{-i})$.

If $\beta_i = \alpha$, then agent i gets one of the above two utilities depending on the leader's decision.

Over-reporting, i.e., $\beta_i > \lambda_i$, is dominated by truthful reporting of λ_i: (1) when $\beta_i > \lambda_i \geq \alpha$, over-reporting and truthful reporting generate the same payoff to agent i: $U(\widehat{g};\lambda_i) -$

$\theta(\widehat{g}, \widehat{b}; \beta_{-i})$; (2) when $\alpha > \beta_i > \lambda_i$, over-reporting and truthful reporting generate the same payoff to agent i: $U(\widehat{g}; \alpha) - \theta(\widehat{g}, \widehat{b}; \beta_{-i})$; (3) when $\beta_i > \alpha > \lambda_i$, by over-reporting, agent i gets $U(\widehat{g}; \lambda_i) - \theta(\widehat{g}, \widehat{b}; \beta_{-i})$, which is less than agent i can get by reporting truthfully (in which case he gets $U(\widehat{g}; \alpha) - \theta(\widehat{g}, \widehat{b}; \beta_{-i})$); (4) when $\beta_i = \alpha > \lambda_i$, agent i gets no better payoff by over-reporting than by truthful-reporting. Similarly under-reporting is also dominated by truthful reporting. Thus, the truthful report is a dominant strategy and the unique undominated strategy for every non-leading agent.

(b) We analyze stage 1

If $g^*(\lambda_N) \leq 0$, it is clear that the unique Nash equilibrium outcome is zero utility for all.

Now we consider the case when $g^*(\lambda_N) > 0$. If all λ_i are zero except for λ_1, it is easy to check that agent 1 would win stage 1 by reporting $g_1 = g^*(\lambda_1), b_1 = b^*(\lambda_1)$ and g^* is produced.

Let $\overline{g} = \max_{-i} g_j$ be the highest bid of the optimal production level by agents other than i and $\overline{b} = \max\{b_j : (g_j, b_j) = (\overline{g}, b_j), j \neq i\}$ be the highest bid of joint (marginal) benefit at $\overline{g}$.

Claim 3. *If $\overline{g} < g^*$ or ($\overline{g} = g^*, \overline{b} < b^*$), player i is better off to win by bidding ($g_i = g^*, b_i = b^*$) than to lose by bidding less than $(\overline{g}, \overline{b})$. If $\overline{g} > g^*$ or ($\overline{g} = g^*, \overline{b} > b^*$), player i is better off to lose the auction (of leadership) in stage 1.*

Proof of Claim 3

If $\overline{g} < g^*$ or ($\overline{g} = g^*, \overline{b} < b^*$), and agent i does not win the auction in stage 1. Agent i is a follower in stage 2. In the unique undominated equilibrium of the sub-game at stage 2, every non-leading agent reveals his type; the leader, $i^* \neq i$, reports his inferred type, $\beta_{i^*} = \widetilde{\lambda}_{i^*}(\overline{g}, \overline{b}; \lambda_{-i^*})$, and produces $\overline{g}$-unit of the public good. Notice that $\beta_{i^*} < \lambda_{i^*}$ and all the followers' inferred types are their true types: $\widetilde{\lambda}_i(\overline{g}, \overline{b}; \beta_{i^*}; \lambda_{-i,i^*}) = \lambda_i$. Hence agent i's payoff $U(\overline{g}; \lambda_i) - \theta_i(\overline{g}, \overline{b}; \beta_{i^*}, \lambda_{-i,i^*}) = U(\overline{g}; \lambda_i) - F_i(\beta_{i^*}; \lambda_{-i^*})$. By bidding (g^*, b^*), agent i can get $U(g^*, \lambda_i) - F_i(\lambda_{i^*}; \lambda_{-i^*})$. Since F is a unit-by-unit cost-sharing rule, there exists a $s_i^g(\lambda_1, ..., \lambda_n)$

for each g such that $F_i(\lambda_{i^*};\lambda_{-i^*}) = \sum_{g=1}^{g^*} s_i^g(\lambda_1,...,\lambda_n)$. Let $F_i(\lambda_N)|_q = \sum_{g=1}^{g=q} s_i^g(\lambda_N)$.

$$\begin{aligned}
U(\overline{g};\lambda_i) - F_i(\beta_{i^*};\lambda_{-i^*}) &\leq U(\overline{g};\lambda_i) - F_i(\lambda_N)|_{\overline{g}} \\
&\quad \text{(because } s^g \text{ satisfies Cross Monotonicity)} \\
&\leq U(g^*,\lambda_i) - F_i(\lambda_{i^*};\lambda_{-i^*}). \\
\text{(because } 0 &\leq s_i^g(\lambda_N) \leq v(g,\lambda_i) \text{ for any } g \in (\overline{g},g^*])
\end{aligned}$$

The first half of the claim is established.

If $\overline{g} > g^*$ or $(\overline{g} = g^*, \overline{b} > b^*)$, by bidding more than $(\overline{g},\overline{b})$, agent i becomes the leader in stage 2. Denote agent i's bid (g_i,b_i). Since all other agents reveal their true types in stage 2, agent i's equilibrium strategy is to report his inferred type: $\widetilde{\lambda}_i(g_i,b_i;\lambda_{-i})$. Note that his inferred type is larger than his true type: $\widetilde{\lambda}_i > \lambda_i$. If agent i chooses to produce g_i-unit of the public good, agent i's cost-share is $C(g_i) - \sum_{j\neq i}\theta_j(g_i,b_i;\lambda_i,\lambda_{-i,j}) = C(g_i) - \sum_{j\neq i} F_j(\widetilde{\lambda}_i,\lambda_{-i})$, which is greater than $F_i(\lambda_N)$. If agent i chooses not to produce, agent i's payoff is negative since he needs to compensate all the other agents.

If in stage 1, agent i chooses to bid less than $(\overline{g},\overline{b})$, agent i will get $U(\overline{g};\lambda_i) - \theta_i(\overline{g},\overline{b};\beta_{i^*},\lambda_{-i^*,i})$, regardless of whether the public good is produced or not, where i^* denotes the leader and $\beta_{i^*} = \widetilde{\lambda}_{i^*}(\overline{g},\overline{b};\lambda_{-i^*})$. Note that agent i's inferred type is the same as his true type: $\widetilde{\lambda}_i(\overline{g},\overline{b};\beta_{i^*};\lambda_{-i,i^*}) = \lambda_i$. Notice that $\beta_{i^*} > \lambda_{i^*}$ and $\overline{g} > g^*$. We have:

$$\begin{aligned}
& U(\overline{g};\lambda_i) - \theta_i(\overline{g},\overline{b};\beta_{i^*},\lambda_{-i^*,i}) \\
= \; & U(\overline{g};\lambda_i) - F_i(\beta_{i^*},\lambda_{-i^*}) \\
\geq \; & U(g^*;\lambda_i) - F_i(\beta_{i^*},\lambda_{-i^*})|_{g^*} \\
& \text{(because } s^g \text{ satisfies No Subsidy for any } g \in (g^*,\overline{g}]) \\
\geq \; & U(g^*;\lambda_i) - F_i(\lambda_{i^*},\lambda_{-i^*}) \\
& \text{(because } s^g \text{ satisfies Cross Monotonicity)}
\end{aligned}$$

Therefore, if $\overline{g} > g^*$ or $(\overline{g} = g^*, \overline{b} > b^*)$, agent i prefers not to win in stage 1. This establishes Claim 3. ■

Theorem 3 follows directly from Claim 3. ■

4 Conclusion

Jackson and Moulin (1992) propose a simple mechanism to implement a large family of cost-sharing rules, including the proportional cost-sharing rule, for a binary public good. In trying to generalize Jackson and Moulin's mechanism into non-binary public goods, we found that the difficulty (of generalizing) lies in the fact that the family of cost-sharing rules considered by Jackson and Moulin (1992) does not always choose allocations within in the core when the public good is not binary. This leads to the questions of what types of cost-sharing rules always pick allocations in the core and how to implement those rules. This paper partly answers those questions. More work should be done to characterize the unit-by-unit version of some common cost-sharing rules.

Appendix

Proof of the Claim 1:

It suffices to show that:

$$\sum_{i\in N}\min\{\sum_{g=1}^{k-1}y_i(g)+u_i(k),\widetilde{H}_i(t_{k-1})\}<C(k). \tag{3}$$

First, we show that (3) holds for $k=2$. Let $T=\{i:u_i(1)<\widetilde{H}_i(t_1)\}$ be the set of agents whose cost-share for the first unit is bounded by their willingness to pay for the good. T might be empty.

We have: $\sum_{i\in T}u_i(1)<c(1)$. Since the marginal cost is increasing and the marginal utility is non-increasing, we must have $\sum_{i\in T}u_i(2)<c(2)$. Therefore:

$$\sum_{i\in N}\min\{y_i(1)+u_i(2),\widetilde{H}_i(t_1)\}\leq\sum_{i\in N}y_i(1)+\sum_{i\in T}u_i(2)<c(1)+c(2)=C(2).$$

Then, we show that if (3) holds for $k=J-1$, then (3) holds for $k=J$.

Let $T=\{i:\sum_{g=1}^{J-2}y_i(g)+u_i(J-1)<\widetilde{H}_i(t_{J-1})\}$. If (3) holds for $k=J-1$, then $t_{J-2}<t_{J-1}$. Therefore, $\sum_{i\in T}u_i(J-1)<c(J-1)$. Since the marginal cost is increasing and the marginal utility is non-increasing, we must have $\sum_{i\in T}u_i(J)<c(J)$. Consequently,

$$\sum_{i\in N}\min\{\sum_{g=1}^{J-1}y_i(g)+u_i(J),\widetilde{H}_i(t_{J-1})\}\leq\sum_{i\in N}\sum_{g=1}^{J-1}y_i(g)+\sum_{i\in T}u_i(J)<C(J-1)+c(J)=C(J),$$

i.e., (3) holds for $k=J$. Therefore, (3) holds for all $k=2,3,...g^*$. $\blacksquare$

Proof of Claim 2:

First, we show that $y_i(1) \geq \{x_i - (U_i(g^*) - U_i(1))\}_+$ for all $i \in N$.

$y_i(1) = \min\{U_i(1), \widetilde{H}_i(t_1)\}$. If $y_i(1) = U_i(1)$, since $x_i \leq U_i(g^*)$ by the requirement of the core, $y_i(1) \geq \{x_i - (U_i(g^*) - U_i(1))\}_+$. If $y_i(1) = \widetilde{H}_i(t_1)$, since $t_1 \geq 1$ by design, we have $y_i(1) \geq H_i^+(1)$. Claim 2 holds for $k = 1$.

Then, we show that if Claim 2 holds for $k = J - 1$, it also holds for $k = J$.

Suppose that Claim 2 holds for $k = J-1$, i.e., $\sum_{g=1}^{J-1} y_i(g) \geq \{x_i - [U_i(g^*) - U_i(J-1)]\}_+ = H_i^+(J-1)$ for all $i \in N$.

$$\sum_{g=1}^{J} y_i(g) = \min\{\sum_{l=1}^{J-1} y_i(l) + u_i(J), \widetilde{H}_i(t_J)\}$$

If $\sum_{g=1}^{J} y_i(g) = \sum_{g=1}^{J-1} y_i(g) + u_i(J)$, then $\sum_{g=1}^{J} y_i(g) \geq H_i^+(J-1) + u_i(J) \geq H_i^+(J)$.

If $\sum_{g=1}^{J} y_i(g) = \widetilde{H}_i(t_J)$, since $t_J \geq J$ by design, we have $\sum_{g=1}^{J} y_i(g) = \widetilde{H}_i(t_J) \geq H_i^+(J)$. Therefore, if Claim 2 holds for $k = J - 1$, it also holds for $k = J$. Consequently, Claim 2 holds for all $k = 1, 2, ..., g^*$. ■

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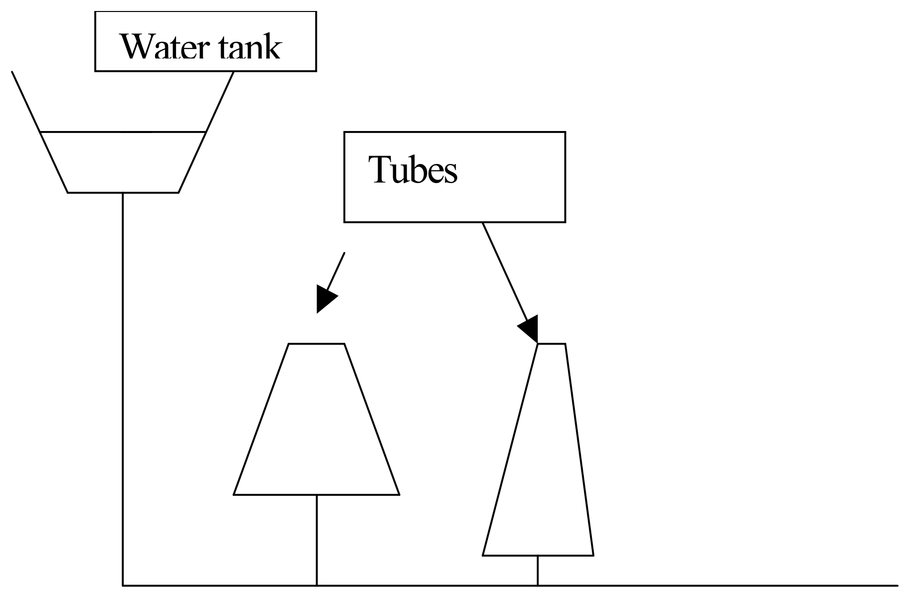

Figure 1: Water Tank and a Tube for Each Agent