Float on a Note*

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Abstract

From 1863-1914, banks in the U.S. could issue notes subject to full collateral, a tax on outstanding notes, redemption of notes on demand, and a clearing fee per issued note cleared through the Treasury. The system failed to satisfy a purported arbitrage condition: the yield on collateral exceeded the tax rate plus the product of the clearing fee and the average clearing rate of notes. The failure is explained by a model in which note issuers choose to issue notes only in trades that produce a low clearing rate (high float), but in which there are diminishing returns to additional note issue.

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1 Introduction

During the last few decades of the nineteenth century and for some years thereafter, many countries had monetary systems in which privately owned banks issued notes under a variety of rules. Friedman and Schwartz [5] describe the U.S. system, the National Banking System (NBS), and point to what they regard as a puzzle. They describe various ways of measuring the rate of return on additional note issue and conclude:

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> Either bankers did not recognize a profitable course of action..., which is hard to accept, or we have overlooked some costs of bank note issue that appeared large to them, which seems much more probable ([5], page 24).

The puzzle was that the yield on the assets that could have been used as backing for note issue, eligible collateral, seemed to be too high. Put differently, the system did not produce an elastic currency at a sufficiently low interest rate—not even elasticity with respect to the yield on eligible collateral. The absence of such elasticity was viewed as the main defect of the NBS—so much so that the title of the act that created the Federal Reserve System, which replaced the NBS, included the phrase, "to furnish an elastic currency."

We provide a new explanation of why the NBS and similar note-issue systems seemed to give rise to unexploited profit opportunities for banks and did not produce elastic currencies. Our theory has two crucial ingredients. First, the profitability of note issue depends on the float on newly issued notes. Second, float depends on banker decisions about where to issue notes: there are unlimited note-placement opportunities, but these produce almost no float and, therefore, are not used; other opportunities provide substantial float and are used, but they are limited in that they exhibit diminishing returns to additional note issue.

Under the NBS, banks with national charters could issue notes under four main restrictions (see, for example, Friedman and Schwartz [5], pp. 20–23): (i) full collateral in the form of government bonds; (ii) a per-period tax on outstanding notes; (iii) redemption of notes into specie (or outside money) on demand; (iv) a clearing fee per issued note that is cleared through the Treasury's clearing system. These rules can be described in more familiar terms as a way of operating a central-bank discount window. The central bank lends at an interest rate equal to the tax rate in (ii) subject to the collateral requirement in (i). Loans take the form of notes that are identified with the borrower—perhaps, by their serial numbers (analogous to notes under the NBS identified by the issuing bank). When notes associated with a given loan are cleared through the central bank's clearing system, the borrowing bank's debt and collateral are debited by the amount cleared, which corresponds to (iii), and a fee, the fee in (iv), is imposed proportional to the amount cleared.

Under such a discount window scheme, strange though it may be, a borrowing bank would be concerned about the float rate implied by different uses of the notes it receives from the central bank. The same is true for the notes a bank issues in a note–issuing system.[1] Although previous researchers did not ignore float, they assumed, explicitly or implicitly, that the float rate implied by a bank's note issue was exogenous to the issuing bank. That led to the view that the system should have produced an upper bound on the yield on collateral given by the tax rate in (ii) plus the product of the clearing fee in (iv) and the average clearing rate (the inverse of the average float rate) of notes. However, as is well known, yields on eligible collateral were generally higher than such a bound (see, [3]).[2]

Because the clearing rate was random, one route to an explanation of the violation of that bound is risk aversion regarding clearing fees. We do not take that route. Our explanation is a model in which the observed average clearing rate is determined by the behavior of note issuers: they face a menu of choices regarding situations in which to issue notes, a menu that displays an inverse association between the float rate and the size of placement opportunities for notes. Faced with that menu, banks issue notes only in situations that give rise to sufficiently high float (a low clearing rate). Therefore, the average clearing rate implied by the model is as low as it is because sufficiently low float-rate placement opportunities are not used. According to the model, the upper bound on the yield of eligible collateral estimated by previous researchers is not valid because it treats the observed average clearing rate as if it applied to all potential placements of notes.

The two main ingredients of our explanation are plausible. The concern about float is dictated by the rules for note issue. The inverse association between the float rate and the size of placement opportunities for notes is consistent with the notion that large placement opportunities would have been available only in organized financial markets. But notes used in such

[1] The idea that float was a relevant concern for an issuer was set out, but not modeled, in [4].

[2] There is data on redemptions that occur through the Treasury's clearing system and on the fee charged for such redemptions. There is no data on redemptions that occur in other ways—over the counter—or on the costs of such redemptions. In [3], the authors infer total redemptions from data on currency clearings for the Federal Reserve System and assume that the fee charged by the Treasury applies to total redemptions. Then, they argue that the implied costs are high enough to bring the presumed bound into equality with the yield on eligible collateral. Others dispute this way of inferring total redemptions and the application of the Treasury fee to total redemptions (see [7] and [9]).

markets would have very quickly been turned in to other banks, banks which had an incentive to clear them through the Treasury's note-clearing system.[3] Indeed, a high float rate and most conceptions of large markets seem incompatible. In order to achieve high float, notes should be offered in trade to people who are infrequently connected to such markets. And, almost by definition, such placement opportunities are limited; they give rise to diminishing returns to additional note issue.[4]

The model contains an extreme version of the inverse association between the float rate and the size of placement opportunities for notes. There are two kinds of placement opportunities. The large (unlimited) opportunity is the use of notes to purchase bonds offered on tap by the government. However, that use of notes is assumed to give rise to redemption in one period and, therefore, is unprofitable given the assumed parameter restrictions. The other use is in pairwise meetings. That use gives rise to a random and higher float rate and is profitable on average, but is subject to diminishing returns.

Although our model is extreme, it suffices to convey our two main messages. One is substantive and concerns how note-issue systems like the NBS could have been amended to produce elastic currencies. As may be guessed on the basis of the above analogy with a discount window system, if the requirement for redemption on demand had been dropped, then the note-issue systems would have produced elastic currencies. The other is methodological and concerns the crucial role of decentralized, small-group trade in our model. Although float was not completely ignored in discussions of note-issue systems, it could not be easily accommodated within models. The competitive general equilibrium model—even amended by inserting real balances as an

[3]Friedman and Schwartz hint at this when they say "An issuing bank ... had no way of identifying banks that returned its notes to the Treasury for redemption; hence its New York City correspondents could do so with impunity." ([5], footnote 8, page 21.)

[4]After seeing an earlier draft of this paper, Bruce Champ came across the following remarks from an interview of a Canadian banker by staff of the National Monetary Commission: "A few years ago some experiments were made with market notes with a view to ascertaining the relative value of the circulation afforded by different classes of accounts. The average time for which the notes were outstanding in the cases to the which the test was applied was as follows (in days): Cheese buyers accounts 63; Grain and milling accounts 36; Cheese factory accounts 34; Railway pay cheques 27; Loan company accounts 7 ([15], page 217)."

In addition, there is a related contemporaneous "observation" for Sweden. It seems to be well-known that note-issuing banks in Stockholm dispatched agents to remote regions in Sweden in order to place notes.

argument of utility functions or by adding cash-in-advance constraints—has no room for float. Float and centralized markets are antithetical concepts. Float requires an absence of connections, while centralized markets provide universal connections. Hence, it is no accident that a matching structure—a structure with very few connections among people—turns out to play a key role in our model.

2 The Model

To show that the above ideas are coherent, we construct a model and show that it has a steady state that displays the paradox concerning note-issue profitability. Many such models could be constructed. Our model stays close to existing literature. As in Cavalcanti and Wallace [1], there are two exogenous groups: note issuers, labeled *bankers*, and others, labeled *nonbanker*s. Each period is divided into two parts: a round of pairwise trade (the small group trade) followed by simultaneous note clearing, the levying of fees, taxation, and bond transactions—all involving the bankers only. (The assumed nonparticipation of nonbankers at the clearing stage is what generates float for notes issued to nonbankers in pairwise meetings.) The background environment for pairwise trade is essentially the version of the Trejos-Wright [12] and Shi [10] models studied by Zhu [14]. To simplify the model, all payments at the note-clearing stage are in the form of a single perishable good so that those payments do not become part of the state of the economy.[5] In addition, there is risk neutrality regarding those payments.

All assets are indivisible and, as in Zhu [14], individual wealth is bounded, but otherwise general. (The indivisibility simplifies the existence argument.) There are three kinds of assets: outside money, government bonds (with exogenous real dividends payable in the form of the single good at the clearing stage), and notes issued by bankers. At the clearing stage, the government makes available to bankers one-period bonds: each bond costs a unit of outside money (or a note) and is a title to one unit of outside money and r amount of the good at the next clearing stage. (The bonds are registered, as opposed to being payable-to-the-bearer, and, therefore, cannot be traded

[5] The assumption that interest, taxes, and fees are real is an adaptation of an analogous assumption in [6]. There, the storage costs for assets that are realized at a date are utility costs that do not affect the state of the economy at the beginning of the next date. Here, the same is true for the interest, taxes, and fees that are realized at a date.

in pairwise meetings.) We consider only equilibria in which nonbankers view all notes and outside money as perfect substitutes. Only bankers can hold bonds and only bankers are subject to lump-sum taxes at the clearing stage that balance the government's budget.

As regards pairwise trade, there are $N \geq 3$ perishable types of goods at each date and a unit measure of each of N specialization types of people. For $n \in \{1, 2, ..., N\}$, a type n person consumes only good n and is able to produce only good $n+1$ (modulo N). Each person maximizes expected discounted utility. For each specialization type, the fraction α are "bankers," while the remainder, the fraction $1-\alpha$, are "nonbankers," where $\alpha \in (0, 1/2)$ and is best thought of as being small.[6] The common period utility function is

$$U(c_1, p_1, c_2, p_2) = u(c_1) - p_1 + (c_2 - p_2), \tag{1}$$

where c_1 is consumption and p_1 is production in pairwise meetings, and c_2 is consumption and p_2 is production of the single good at the clearing stage.[7] Goods are perishable at each stage and $c_2 \equiv p_2 \equiv 0$ for each nonbanker. There is a common discount factor $\beta \in (0, 1)$. The function $u : \mathbb{R}_+ \to \mathbb{R}$ is strictly concave, strictly increasing, continuously differentiable, and satisfies $u(0) = 0$, $u'(\infty) = 0$, and $u'(0)$ sufficiently large. We assume that each banker always meets a nonbanker, that a nonbanker meets a banker with probability $\lambda = \alpha/(1-\alpha)$, and that a nonbanker meets another nonbanker with probability $1-\lambda$. In other respects, meetings are random.

Prior to pairwise meetings, each banker starts with the following balance sheet. The assets are outside money, y_m, and bonds, y_b; the liabilities are own-notes held by the government, y_g, and own-notes held by nonbankers, y_n. All four items are measured in units of outside money.

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[6] There were thousands of National Banks under the NBS.

[7] If the clearing stage included a market in which bankers could trade accumulated wealth for the clearing-stage good as in Lagos and Wright [8], then the model would have a degenerate distribution of wealth among bankers. We exclude such trade because it does not seem descriptive of actual note-issuing systems and because our assumption that payoffs at the clearing stage are in goods is solely a modeling convenience designed to prevent such payoffs from becoming part of the state of the economy. Moreover, even if we included trade among bankers at the clearing stage, such trade would not produce the simplicity it does in [8]; the distribution of wealth among nonbankers would remain endogenous and nondegenerate and the distribution of wealth between bankers and nonbankers would remain endogenous.

Banker Balance Sheet	
Assets	Liabilities
y_m (outside money)	y_g (own-notes held by the government)
y_b (bonds)	y_n (own-notes held by nonbankers)

We distinguish between the two kinds of notes because they have different consequences for float. A banker's wealth is $y_m + y_b - y_g - y_n$. Each nonbanker holds some money, the sum of outside money and notes held. Letting z denote the wealth of a person, either a banker or a nonbanker, we impose the restriction $z \in \{0, 1, ..., Z\} \equiv \mathbf{Z}$. The upper bound is imposed to achieve compactness. The lower bound for bankers mimics the full-backing feature of NBS notes.

In meetings between nonbankers, we assume take-it-or-leave-it offers by buyers (consumers). In meetings between bankers and nonbankers, we assume that the banker always makes a take-it-or-leave-it offer—whether the banker is a buyer (consumer), a seller (producer), or is neither. Among other things, this permits the banker to exchange own-notes for the notes of other bankers or outside money on a one-for-one basis. Such swaps of monies, among which the nonbanker is indifferent, generate one source of note clearings.[8] Throughout, we allow lotteries in trades.

At the clearing stage, bonds mature (turn into outside money and pay a real coupon), a tax is paid on own-notes outstanding (where the tax base is total note liabilities, notes are cleared (which matters because each banker is charged a real fee, lost to nature, proportional to the amount of its notes that are cleared), and a lump-sum tax is levied. We assume that all notes held by the government are cleared and that all notes held by bankers are cleared.[9] Thus, after clearing, no banker holds other bankers' notes, which

[8] The conclusion that a nonbanker's wealth when leaving a meeting with a banker is entirely in the form of that banker's notes does not depend on the assumption that bankers make take-it-or-leave-it offers in all meetings. Given that nonbankers are indifferent about which notes are held and that bankers are not, that conclusion will follow from any bargaining outcome that produces an outcome in the pairwise core for the meeting.

[9] If notes submitted for clearing by bankers were a choice variable, then one equilibrium would have all notes submitted. We simply assume that that is the action taken. In fact, in our equilibrium, although bankers are indifferent about redeeming such notes, if they did not redeem, then they would certainly want to use such notes to buy bonds. If banks did that, then average float for notes issued in pairwise meetings would increase by one period. Such an increase would lead to only a slight modification in our arguments.

is why such an entry does not appear in the balance sheet. Then bankers buy new bonds (or outside money) from the government, either with outside money or own-notes.

Although all our claims for this model are about steady states, a few remarks are in order about how the economy evolves from a somewhat special initial condition. Consider an initial condition prior to pairwise meetings in which each person, banker or nonbanker, holds only some outside money, except that each banker has, in addition, Z own-notes. (Under the NBS, the government printed all the notes, which, however, were identified by the issuing bank.) As is standard, own-notes held by a banker are neither an asset nor a liability. We assume that the banker starts with Z own-notes because that is the maximum feasible quantity that might be traded in a pairwise meeting. The total amount of outside money, expressed as the average amount per specialization type, is $\bar{z}$. As the economy evolves, average wealth per specialization type remains $\bar{z}$; that is, in our model wealth gets passed around among people, but is not created. The only assumptions we make about $\bar{z}$ and $Z/\bar{z}$ are that both are sufficiently large—so that neither the indivisibilitity of assets nor the bound is too severe.

At the clearing stage, each banker shows the government how many own-notes are still in the banker's possession and the tax is levied on the difference between the amount so far created and that amount. In addition, the banker must demonstrate satisfaction of the nonnegative wealth constraint. Because liabilities consist of notes outstanding, either held by the government or held by nonbankers, the liabilities can be identified. Therefore, satisfaction of the nonnegative wealth constraint is demonstrated by the banker showing sufficient assets. To assure that each banker has enough own-notes at the end of the clearing stage, we can suppose that each banker's stock of own-notes is augmented to bring it up to Z. We assume that unlimited own-notes are available for the purchase of bonds or additional outside money at the clearing stage.

We do not make explicit the penalty for failing to meet the nonnegative wealth constraint. We simply assume that it is large enough to prevent the constraint from being violated. Nor do we make explicit the penalty for failing to pay the lump-sum tax that is levied to balance the government's budget.

The main assumption we make is

$$r - \rho < \tau < \frac{\beta}{1 + \beta\lambda}(r - \lambda\rho), \tag{2}$$

where r is the bond coupon payment, τ is the tax per outstanding note, and ρ is the fee per note cleared.[10] The first inequality is consistent with most of the data on interest rates and the clearing fee for the NBS episode if a period in the model is something like a week or less.[11] It implies that banks do not use own-notes to buy bonds. As for the second inequality, we show that there is a steady state in which λ, the probability that a note held by a nonbanker "meets" a banker, is the average clearing rate. (It is also an upper bound on the average clearing rate.) The second inequality is equivalent to

$$\tau < \sum_{t=1}^{\infty} \beta^t (1 - \lambda)^{t-1} [\lambda(r - \tau - \rho) + (1 - \lambda)(r - \tau)], \tag{3}$$

which can be given the following interpretation.[12] Issuing a note in a pairwise meeting in exchange for another note costs τ, but can be used to acquire a bond at the following clearing stage. Then there are two possibilities at the clearing stage that follows that one: with probability λ the own-note is cleared and the payoff is $r - \tau - \rho$; with probability $1 - \lambda$ the own-note is not cleared and the payoff is $r - \tau$. Conditional on not being cleared then, the same two possibilities arise at the next clearing stage; and so on. Thus, the second inequality makes such note issue profitable and with λ being the average clearing rate is the paradox we set out to explain.

Our nonnegative wealth constraint on bankers is a bit weaker than the NBS constraint that a bank have bonds deposited that match all the own-notes that it receives from the Treasury. And, although the model's bankers are most straightforwardly viewed as issuers of transferable trade-credit IOUs under some constraints, there is an interpretation that makes them more like NBS banks. If each NBS bank issued notes in the form of a loan to a single borrower, then each banker in the model could be viewed as a consolidation of the NBS bank and its borrower. That interpretation is sensible because

[10] Sufficient conditions for the satisfaction of (2) are λ close to 0 and β close to 1—sufficient, that is, for making the left-most term in (2) less than the right-most term.

[11] As noted in [4, p.356], the Treasury's clearing fee was "sufficient to offset more than two weeks of interest at 2% per year."

[12] We are indebted to a referee for pointing out this interpretation of the second inequality in (2).

an NBS bank had to be concerned about the float rate on its notes and, therefore, had to be concerned about how its customers who borrowed in the form of notes used those notes.

One extreme aspect of our model is that bankers have little control over their opportunities for high-float placements; there is one pairwise meeting per date. While extreme, this specification is not missing anything significant. A model in which a bank could choose a "search intensity" that allows it at a cost to increase the frequency of pairwise meetings would have similar implications provided the cost of higher search intensity is sufficiently convex. Such a model would add a margin that displays smoothly diminishing returns from additional note issue. However, our model already has such margins coming from meetings in which a banker is either a consumer or a producer.[13] Therefore, adding another margin, while seemingly realistic, does not seem worthwhile.

3 Definition of a Steady State

We show that there exists a steady state with several features. First, $y_m = y_g = 0$ for every banker. That is, prior to pairwise meetings, each banker has a single asset, bonds, and a single liability, notes held by nonbankers. Second, when a banker meets a nonbanker holding wealth z, which with probability one is in the form of other bankers' notes, the banker replaces that wealth with own-notes. That, in turn, implies that those notes of other bankers get cleared at the clearing stage. Such behavior implies that the expected number of clearings of own-notes is λy_n. However, to demonstrate that there is such a steady state, we have to begin with a somewhat general definition, one that does not impose that behavior.

A steady state consists of (w, v, π), where, $w : \mathbf{Z} \to \mathbb{R}_+$ is a value function defined on individual nonbanker's wealth and $v : \mathbf{Y} \to \mathbb{R}$, is a value function defined on individual banker balance sheets, where

$$\mathbf{Y} = \{y = (y_m, y_b, y_g, y_n) \in \mathbb{Z}_+^4 : 0 \le y_m + y_b - y_g - y_n \le Z\}, \qquad (4)$$

and $\pi = (\pi_n, \pi_b)$ is a pair of probability measures over wealth where π_i :

[13] If individual wealth holdings are limited to the set $\{0, 1\}$, then there is less scope for such diminishing returns. In addition, a unit upper bound assumption in our model does not simplify the model much. Even in that case, the distribution of wealth between bankers and nonbankers is endogenous (see, for example, [13]).

$\mathbf{Z} \to \mathbb{R}_+$ and $\pi_n(z)$ $[\pi_b(z)]$ is the measure of nonbankers (bankers) of each specialization type who have wealth z. All of these pertain to the start of a date prior to pairwise meetings. Notice that we are not making the distribution of bankers over elements of $\mathbf{Y}$ a part of the description of a steady state. We can get by without describing that distribution because the behavior that affects future wealth turns out to depend only on current wealth, not on its composition.

Our definition of a steady state is standard for heterogeneous-agent models: a steady state is a fixed point of a mapping from beginning-of-period wealth distributions and end-of-period value functions to beginning-of-period value functions and end-of-period wealth distributions, a mapping which is defined using the optimizing take-it-or-leave-it offers in pairwise meetings and the optimal bond purchases for bankers at the clearing stage. However, because there are two stages and different kinds of pairwise meetings and because we permit lotteries, the definition is lengthy. Readers who wish to skip the details of the definition can without loss of continuity go directly to the next section where we present and discuss the result.

We start with choices and payoffs in pairwise meetings between nonbankers. Let Q be an upper bound on output that we describe in the existence proof. For $(z, z') \in \mathbf{Z}^2$ (where z is the wealth of the buyer and z' is that of the seller), we let $\Gamma(z, z', w)$ be a set of probability measures on $[0, Q] \times \{\max\{0, z + z' - Z\}, ..., z\}$, given by

$$\Gamma(z, z', w) = \{\sigma : E_\sigma[-q + \beta w(z + z' - z'')] \geq \beta w(z')\}. \tag{5}$$

Here, E_σ denotes the expectation with respect to σ, the arguments of which are (q, z'')—q being output and z'' being the post-trade wealth of the buyer. Then,

$$f(z, z', w) = \max_{\sigma \in \Gamma(z, z', w)} E_\sigma[u(q) + \beta w(z'')] \tag{6}$$

is the payoff for a nonbanker buyer with z who meets a nonbanker seller with z'.

For bankers, we start with the clearing-stage stage. A banker with y prior to pairwise meetings and with pairwise trade realization (d_a, d_l), where d_a is the addition to the banker's assets and d_l is the addition to the banker's liabilities, and with clearing-stage redemptions from notes previously held by nonbankers x chooses (y'_m, y'_b, y'_g) to maximize

$$v(y'_m, y'_b, y'_g, y_n + d_l - x) \tag{7}$$

subject to

$$y'_m + y'_b \leq y'_g + y_m + y_b + d_a - x - y_g. \tag{8}$$

The payoff for such a banker is

$$\begin{aligned} g(d_a, d_l, y, x, v) &= ry_b - \rho(x + y_g) - \tau(y_g + y_n + d_l) - \iota + \\ &\quad \beta \max v(y'_m, y'_b, y'_g, y_n + d_l - x), \end{aligned} \tag{9}$$

where ι is the lump-sum tax and where the maximization is subject to (8).

Now, we turn to banker choices in pairwise meetings. For $(z, y) \in \mathbf{Z} \times \mathbf{Y}$, where z is the wealth of the nonbanker and y is the portfolio of the banker, we let

$$\begin{aligned} d(z, y) &= \{(d_a, d_l) \in \mathbb{Z} \times \mathbb{Z}_+ : -y_m \leq d_a \leq z, \\ \tilde{y} - Z &\leq d_l - d_a \leq \min\{Z - z, \tilde{y}\}, \end{aligned} \tag{10}$$

where d_a is the addition to the banker's assets and d_l is the addition to the banker's liabilities in the meeting, and where $\tilde{y} = y_m + y_b - y_g - y_n$, the wealth implied by y. (The second two inequalities impose the bounds on wealth.) Notice that if d_a is never negative, which turns out to be the case, then the restriction $-y_m \leq d_a$ can be omitted and $d(z, y)$ depends on y only by way of the wealth implied by y. We define $\Gamma_i(z, y, w)$, $i = 1, 2, 3$, sets of probability measures on $[0, Q] \times d(z, y)$, by

$$\Gamma_1(z, y, w) = \{\sigma : E_\sigma[-q + \beta w(z - d_a + d_l)] \geq \beta w(z)\}, \tag{11}$$

$$\Gamma_2(z, y, w) = \{\sigma : E_\sigma[u(q) + \beta w(z - d_a + d_l)] \geq \beta w(z)\}, \tag{12}$$

and

$$\Gamma_3(z, y, w) = \{\sigma : E_\sigma \beta w(z - d_a + d_l)] \geq \beta w(z)\}, \tag{13}$$

where E_σ again denotes the expectation with respect to σ and where the arguments of σ are now (q, d_a, d_l). (Notice that the Γ_i depend on y only by way of the dependence of the set $d(z, y)$ on y.) Then, the banker payoffs are

$$f_1(z, y, w, v) = \max_{\sigma \in \Gamma_1(z,y,w)} E_\sigma[u(q) + E_x g(d_a, d_l, y, x, v)] \tag{14}$$

$$f_2(z, y, w, v) = \max_{\sigma \in \Gamma_2(z,y,w)} E_\sigma[-q + E_x g(d_a, d_l, y, x, v)] \tag{15}$$

$$f_3(z, y, w, v) = \max_{\sigma \in \Gamma_3(z,y,w)} E_\sigma[E_x g(d_a, d_l, y, x, v)], \tag{16}$$

where E_x denotes expectation w.r.t. the distribution of x, and where g is given by (9). Here, $f_1(z, y, w, v)$ is the payoff for a banker buyer with y who meets a nonbanker seller with z, $f_2(z, y, w, v)$ is the payoff for a banker seller with y who meets a nonbanker buyer with z, and $f_3(z, y, w, v)$ is the payoff for a banker with y who meets a nonbanker with z in a no-coincidence meeting. As we will see, the form of g will be such that the only aspect of the distribution of x that is needed to compute the expectation w.r.t. x is the mean of that distribution. That is why we can get by without including the distribution of clearings as part of the steady state.

We denote the set of maximizers in (6) by $\widetilde{\Delta}(z, z', w)$ and those in (14), (15), and (16) by $\widetilde{\Delta}_1(z, y, w, v)$, $\widetilde{\Delta}_2(z, y, w, v)$, and $\widetilde{\Delta}_3(z, y, w, v)$, respectively. Because it can be shown that all the maximizers are degenerate in q, in what follows, for a maximizer, we denote the maximizing q by $\hat{q}$.

Now, using $\widetilde{\Delta}$ and the $\widetilde{\Delta}_i$, we define sets of optimal end-of-trade wealth that are needed to define the steady state conditions for π. For a meeting between a nonbanker buyer with z and a nonbanker seller with z', we define $\Delta(z, z', w)$, a set of probability measures on $\mathbf{Z}$, by

$$\Delta(z, z', w) = \{\delta : \delta(.) = \widetilde{\delta}(\hat{q}, .),\ \widetilde{\delta} \in \widetilde{\Delta}(z, z', w)\}.$$

Then we define $\Omega(w, \pi)$, a set of probability measures on $\mathbf{Z}$, by

$$\begin{aligned} \Omega(w, \pi) \;=\; & \{\omega : \omega(z) = \textstyle\sum_{(z',z'')} \pi_n(z')\pi_n(z'')[\delta(z) + \delta(z' - z + z'')], \\ & \text{for } \delta \in \Delta(z', z'', w)\}, \end{aligned} \tag{17}$$

where, as a convention, $\delta(x) = 0$ if $x \notin \mathbf{Z}$.

For a banker with y who meets a nonbanker with z', for $i = 1, 2, 3$, and for $(z, z', y) \in \mathbf{Z}^2 \times \mathbf{Y}$, we let $z^{-1}(z', y) = \{(d_a, d_l) : z' - d_a + d_l = z\}$. (That is, $z^{-1}(z', y)$ is the set of asset trades that leave the nonbanker with wealth z.) Then we define $\Delta_i(z', y, w, v)$, a subset of probability measures on $\mathbf{Z}$ by

$$\begin{aligned}\Delta_i(z', y, w, v) &= \{\delta : \delta(z) = \textstyle\sum_{z^{-1}(z',y)} \widetilde{\delta}(\hat{q}, d_a, d_l), \\ \text{for } \widetilde{\delta} &\in \widetilde{\Delta}_i(z', y, w, v)\}.\end{aligned}$$

Let μ be an arbitrary measure on $\mathbf{Y}$ that is consistent with π. We define $\Theta(w, v, \pi)$, a set of probability measures on $\mathbf{Z} \times \mathbf{Z}$ by

$$\begin{aligned}\Theta(w, v, \pi) &= \{(\theta_n, \theta_b) : \\ \theta_n(z) &= \frac{1}{N} \textstyle\sum_{(z',y)} \pi_n(z')\mu(y)(\delta_1 + \delta_2 + (N-2)\delta_3)(z), \\ \theta_b(z) &= \frac{1}{N} \textstyle\sum_{(z',y)} \pi_n(z')\mu(y)(\delta_1 + \delta_2 + (N-2)\delta_3)(z' - z + \tilde{y}), \\ \text{for } \delta_i &\in \Delta_i(z', y, w, v)\}, \end{aligned} \tag{18}$$

where $\tilde{y} = y_m + y_b - y_g - y_n$ and where, as a convention, $\delta_i(x) = 0$ if $x \notin \mathbf{Z}$.

Now we can complete the conditions for a steady state. The value function w must satisfy

$$w(z) = [\lambda + \frac{(1-\lambda)(N-1)}{N}]\beta w(z) + \frac{(1-\lambda)}{N} \sum \pi_n(z') f(z, z', w), \tag{19}$$

and the value function v must satisfy

$$v(y) = \frac{1}{N} \sum \pi_n(z)[f_1(z, y, w, v) + f_2(z, y, w, v) + (N-2) f_3(z, y, w, v)]. \tag{20}$$

The measures $\pi = (\pi_n, \pi_b)$ must satisfy

$$\sum_z z[\alpha\pi_b(z) + (1-\alpha)\pi_n(z)] = \bar{z}, \tag{21}$$

and

$$\pi = (\lambda\theta_n + (1-\lambda)\omega, \theta_b) \text{ for some } (\omega, (\theta_n, \theta_b)) \in \Omega(w, \pi) \times \Theta(w, v, \pi). \tag{22}$$

Definition 1 *A steady state is* (w, v, π) *that satisfies* (19) – (22).

We have chosen to omit from the definition the condition that the lump-sum tax, ι, is such as to achieve government budget balance; namely,

$$\tau(1-\alpha)\sum_z z\pi_n(z) + \iota\alpha = r\bar{z}. \tag{23}$$

Because ι does not affect behavior, we can simply insert into (9) the ι that satisfies (23) for a (w, v, π) that satisfies definition 1. Notice that (23) is written under the assumption that the clearing fee covers a real resource cost and, therefore, is not a source of net revenue for the government. If it were, then τ in this expression would be replaced by $\tau + \lambda\rho$ and nothing else would be affected.

4 Existence of a Steady State

As noted above, we prove that there exists a steady state with some properties. In particular, the value function for bankers defined on portfolios implies the following corner solutions for portfolios and trades: prior to pairwise trade, all banker assets are in the form of bonds, no own-notes are held by the government, and bankers make full use of arbitrage opportunities in pairwise meetings with nonbankers.

Proposition 1 *There exists a steady state (w, v, π) in which w is strictly increasing and strictly concave, $h(\cdot) \equiv v(0, \cdot, 0, 0)$ is strictly concave and satisfies $h(z+1) - h(z) \geq \frac{r}{1-\beta}$. Also, v satisfies*

$$v(y_m, y_b, y_g, y_n) = v(0, y_m + y_b, y_g, y_n) - ry_m, \tag{24}$$

$$v(y_m, y_b, y_g, y_n) = v(y_m, y_b - y_g, 0, y_n) - (\rho + \tau - r)y_g, \tag{25}$$

$$v(y_m, y_b, y_g, y_n) = v(y_m, y_b + s, y_g, y_n + s) - \eta s, \tag{26}$$

where $\eta = \frac{r-\lambda\rho-\tau}{1-\beta+\lambda\beta}$. Finally, π satisfies

$$0 < (1-\alpha)\sum_z \pi_n(z)z < \bar{z}. \tag{27}$$

Condition (24) says that the only consequence for a banker of having some money prior to pairwise meetings rather than having all assets in the form of bonds is the implied loss of interest payments. Condition (25) says that the only consequence of having issued notes to buy bonds is the loss implied by the first inequality in (2). Condition (26) says that the only consequence of s additional bonds and s additional notes held by nonbankers is the gain implied by the second inequality in (2). Inequality (27) says that average wealth for both bankers and nonbankers is positive. If nonbankers have no wealth, then they are in autarky. If bankers have no wealth, then there is no goods trade between bankers and nonbankers.

Our existence proof has several steps. The proof is in the Appendix and is separated into lemmas. The general idea is to show that the mapping implicit in the definition of a steady state has a fixed point that satisfies the properties in Proposition 1. Therefore, some of the proof involves showing that the mapping preserves those properties. Relative to the arguments in Zhu [14], the new parts of the argument involve showing that the mapping preserves (24)-(26), which is established by Lemma 2, and that both bankers and nonbankers have positive average wealth. As the proof of Lemma 2 shows, in a steady state with a v that satisfies the properties in the proposition, no banker holds outside money or uses own-notes to buy outside money or bonds from the government ($y_m = y_g = 0$).[14] Also, each banker replaces all the notes held by the nonbanker who is met with own-notes, from which we conclude that a banker with y_n outstanding prior to pairwise meetings experiences average clearings equal to λy_n.

As regards the distribution of individual bankers by size in terms of gross assets, our steady state determines only averages: from the above properties and (21), average bond holding per banker is $\bar{z}/\alpha$ and average banker liability in the form of notes held by nonbankers is $\sum_z \pi_n(z)z/\alpha$.

We also know the average clearing-stage payoff (to bankers). It is negative and is equal to the average of clearing fees, because the clearing fee is lost to nature. If the clearing fee were a source of net revenue for the government, then the average afternoon payoff would be zero. Obviously, these conclusions depend on the assumption that a lump-sum tax is levied only on bankers. Under our assumptions, the lump-sum tax is positive. If the lump-sum tax were imposed on the entire population and if the measure of bankers were small, then the average afternoon payoff to bankers would be positive. After all, bankers hold their wealth in the form of bonds, while nonbankers hold non-interest bearing notes. If everyone pays the lump-sum tax that finances

[14] We are indebted to Guido Lorenzoni for pointing out an error in an earlier version of the lemma.

interest, then the bankers would gain.

One should not, however, conclude that bankers are on average worse off than nonbankers under our scheme. After all, bankers get to make take-it-or-leave-it offers in all meetings. Average nonbanker welfare is

$$W = \sum_z \pi_n(z) w(z). \tag{28}$$

Average banker welfare can be expressed in terms of the objects in our steady state by making use of (26). By (26),

$$v(0, y_a, 0, y_n) = v(0, z, 0, 0) + \eta y_n, \tag{29}$$

where $z = y_a - y_n$, the banker's wealth. It follows that average banker welfare is

$$V = \sum_z [\pi_b(z) v(0, z, 0, 0) + (\frac{1-\alpha}{\alpha}) \eta \pi_n(z) z], \tag{30}$$

where the second term within the summation comes from market-clearing; namely, equating the average note liability of bankers to the average notes held by nonbankers. (The distribution we use to compute average banker liability is described below.)

If we view entry as a once-for-all decision (before portfolios are assigned) and if we are permitted to assume any distribution of one-time utility costs of entry over the entire population, then we can construct one that makes α the fraction who choose to be bankers. Thus, for $x \in \mathbb{R}$, let $H(x)$ denote the fraction of each specialization type whose utility cost of entry into being a banker does not exceed x. If H is strictly increasing and $H(V - W) = \alpha$, then the fraction α choose to be bankers.[15]

[15] The point of view behind this one-time entry decision is the following. Aside from one-time utility costs of entry, people are identical prior to entry and prior to being assigned portfolios. A person who chooses to be a nonbanker gets expected utility W: the person is assigned wealth z with probability $\pi_n(z)$. A person who chooses to be a banker gets V via the following construction of initial banker portfolios.

Let $x = (1/\alpha - 1) \sum \pi_n(z) z$. Notice that $x > 0$. Now let θ satisfy $\theta[x] + (1-\theta)([x]+1)$, where $[x]$ is the largest integer that does not exceed x. Then a person who chooses to be a banker is assigned $(y_b, y_n) = ([x] + z, [x])$ with probability $\theta\pi_b(z)$ and $(y_b, y_n) = ([x] + 1 + z, [x] + 1)$ with probability $(1-\theta)\pi_b(z)$. Although this distribution of portfolios does not persist, it can serve as an initial condition because it is consistent with the wealth measure π_b and with market clearing.

Previous investigators were not concerned about entry. Given that they were not able to reconcile the second inequality in (2) with finite profits for an existing bank, it was premature to consider entry. Because we reconcile (2) with finite profits for an existing bank, our model opens the way for more elaborate models of entry and exit.[16]

In our model, r, τ, and ρ are measured in goods per note. In the data for the NBS episode, the puzzle is stated in terms of interest rates. To convert the inequalities in (2) into interest-rate units, we need only divide through by the average goods value per note. Obviously, such a restatement leaves the inequalities intact. Moreover, because the trades of goods for notes is determined by the objects in our steady state, both total output (real GDP) and the nominal value of total output (nominal GDP) are implied by a steady state. Therefore, a total output deflator is determined and that along with the average note holding implies an average goods value per note.

In addition, we have the freedom to determine realistic magnitudes for interest rates computed as just described. Consider different magnitudes for (r, τ, ρ) in $\mathbb{R}^3_+$ along a ray through the origin determined so that inequalities (2) hold. As (r, τ, ρ) gets large, the average goods value per note does not get proportionally large because the value function v is bounded independently of (r, τ, ρ). It follows that the interest rate computed as just described gets large as (r, τ, ρ) gets large. As (r, τ, ρ) gets small, the value of a note in a Proposition-1 steady state is bounded away from zero. It follows that the interest rate goes to zero as (r, τ, ρ) gets small. Those two features suggest that we are free to choose (r, τ, ρ) in such a way as to get any magnitude for the steady-state interest rate. Moreover, suppose we consider economies with different magnitudes for $\bar{z}$ for fixed $Z/\bar{z}$. These economies, in effect, have different degrees of asset indivisibility: the larger is $\bar{z}$, the less asset indivisibility there is. Because r, τ, and ρ are in goods per unit of money, it does not make sense to hold them fixed for economies with different $\bar{z}$'s. Therefore, when we assume, as we do in part of the existence proof, that (r, τ, ρ) is sufficiently small, we are, in effect, assuming that assets are sufficiently divisible.

Finally, we want to compare our model to closely related models. First, although the division of agents into bankers and nonbankers is borrowed

[16]If the lump-sum tax were levied on everyone, then no banker would subsequently choose to switch to being a nonbanker if that choice were offered. Bankers face the same probabilities of being consumers and producers as do nonbankers and have more freedom to choose trades; in particular, they can emulate the actions of nonbankers. Of course, nonbankers would then want to switch to being bankers if that were permitted.

from [1], the models are very different. In [1] bankers have known histories. Those histories, by way of threatened punishment, permit the mechanism designer to partially control banker behavior. In particular, that allows outcomes in which the possibility of issuing notes frees banker spending for consumption from the banker's recent acquisition of financial wealth. That can happen because financial wealth is not constrained to be nonnegative in [1]. If there were such a constraint, then the main result in [1]—namely, that note-issue implements strictly more outcomes than does a fixed stock of outside money—would not hold. In the current model, the nonnegative wealth constraint rules out any such role for note issue. Indeed, it is hard to avoid the conclusion that the full-backing requirement under the NBS ruled out any such role of notes in the actual NBS economy. The backing requirement also makes our model quite different from that in [2]. There, only sufficiently high redemptions, which are assumed to give rise to bank failure, prevent over-issue, and, thereby, permit there to be an equilibrium with note issue.

In a complicated way, the role of note issue in the current model is to pay interest on wealth (or money) held by bankers—interest financed in part by lump-sum taxes. However, the model is not equivalent to a model without note issue in which bankers simply get an additive period-utility of r per unit of money held. In such a model, r would be an opportunity cost for a banker who surrenders some wealth for consumption in a pairwise meeting. In our model, a banker who surrenders the same wealth does not sacrifice r. The banker issues notes and those notes are not redeemed immediately. Therefore, the banker surrenders less than r per note issued. In effect, the banker uses the opportunity to consume to engage in some additional arbitrage. Hence, the goods trade is not the same as in a model in which bankers get period-utility of r per unit of money held.

5 Concluding Remarks

As we noted at the outset, there seem to be two possible explanations of the puzzle described by Friedman and Schwartz [5]. One appeals to risk aversion regarding clearing costs, while the other explains how the observed average float rate arises from the behavior of note issuers. These are not mutually exclusive explanations. We omitted the first and showed that the second is sufficient to explain the puzzle. Adding risk aversion would presumably only reinforce the result. However, adding risk aversion greatly complicates the model.

If there is risk aversion concerning clearing fees, then moments of the distribution of note clearings higher than the mean would matter for bankers. And those higher moments depend not only on the total stock of own-notes outstanding, but also on how that stock is distributed among nonbankers: widely dispersed holdings of a given stock imply a smaller variance of clearings than bunched holdings. But that, in turn, implies that the history of a banker's note issues matters because that history influences how widely dispersed is a given stock of outstanding notes. At a minimum, therefore, dealing with risk aversion greatly increases the dimensionality of the state space.

With risk neutrality regarding clearing costs, bankers would like to issue an unlimited quantity of notes that are subject to the steady-state average clearing rate, λ. The model would not have an equilibrium if a banker could do that. Therefore, a critical feature of the model is that the clearing rate λ is available only for notes issued in pairwise meetings, meetings in which the opportunity to arbitrage is limited by the wealth held by the nonbanker in the meeting. That is the model's representation of the general idea that opportunities to place notes that generate high float are limited.

As noted in the introduction, a major defect of the NBS was considered to be its failure to produce an elastic currency—its failure, that is, to bound nominal interest rates at a low level. Our steady state is consistent with the inelasticity concern about the NBS because there is a steady state for all bond coupons satisfying (2). In the model, a bound on r, a bound equal to the tax rate τ, can be achieved by the simple step of eliminating the clearing fee ρ—either by fincancing the clearing cost by lump-sum taxes rather than by the user fee or by assuming that there is no clearing cost. If $\rho = 0$, then $r \leq \tau$ is necessary for equilibrium.[17] However, this literal interpretation of ρ is too restrictive. Even without an explicit fee of the sort that existed under

[17] If $r > \tau$, then there is no solution to the clearing-stage problem of bankers; they want to buy an unlimited quantity of bonds with own-notes. If $\rho = 0$, then there are no choices for the other parameters consistent with even a weak inequality version of the assumption in (2). That is, if $\rho = 0$, then even $r = \tau$ violates the weak version of the second inequality in (2). Therefore, we have not established existence under $\rho = 0$. Indeed, we suspect that our kind of steady state does not exist if $\rho = 0$ even if $r = \tau$. Under our assumed timing and $r = \tau$, notes would not be issued in exchange for the notes of others in pairwise meetings because there is a tax due for the issue period that has no offset in interest earnings.

the NBS, a bank that is forced to redeem its notes on demand incurs costs of adjusting its portfolio. Therefore, it would seem that elimination of the redemption requirement was necessary for an elastic currency. The analogy with a discount-window procedure set out in the introduction suggests that that would have been a workable system.

We end with a comment on the role of models with pairwise meetings in monetary economics. Those models were formulated in order to represent the descriptions of absence-of-double-coincidence difficulties that have for centuries been part of discussions of the role of money. However, to be truly valuable, such models ought to do more than that. We have shown that versions of such models help resolve long-standing puzzles concerning monetary systems in which banks were permitted to issue notes.

6 Appendix

We begin with some notation. Let Q be the unique positive solution to $N(1-\beta)Q = u(\beta Q)$. Let $\mathbf{W}$ be the set of nondecreasing and concave functions $w : \mathbf{Z} \to [0, Q]$. Let $\mathbf{V}$ be the set of functions $v : \mathbf{Y} \to [\frac{-\iota}{1-\beta}, Q + \frac{rZ}{1-\beta}]$ with v nondecreasing and concave in its second argument and with

$$\begin{aligned} v(y_m, y_b, y_g, y_n) &= v(0, y_m + y_b, y_g, y_n) - r y_m, & (31)\\ v(y_m, y_b, y_g, y_n) &= v(y_m, y_b - y_g, 0, y_n) - (\rho + \tau - r) y_g, & (32)\\ v(y_m, y_b, y_g, y_n) &= v(y_m, y_b + s, y_g, y_n + s) - \eta s, & (33) \end{aligned}$$

where $\eta = \frac{r-\lambda\rho-\tau}{1-\beta+\lambda\beta}$. (That is, $v \in \mathbf{V}$ satisfies most of the conclusions in the proposition except that strict properties are replaced by their weak counterparts to make $\mathbf{V}$ closed.)

Next, we formally define the mapping implied by the definition of a steady state. Let the mapping Φ_w on $\mathbf{W} \times \mathbf{V} \times \mathbf{\Pi}$ be defined

$$\Phi_w(w, v, \pi)(z) = [\lambda + \frac{(1-\lambda)(N-1)}{N}]\beta w(z) + \frac{(1-\lambda)}{N} \sum \pi_n(z') f(z, z', w).$$

Let the mapping Φ_v on $\mathbf{W} \times \mathbf{V} \times \mathbf{\Pi}$ be defined by

$$\Phi_v(w, v, \pi)(y) = \frac{1}{N} \sum \pi_n(z)[f_1(z, y, w, v) + f_2(z, y, w, v) + (N-2) f_3(z, y, w, v)]. \quad (34)$$

Let the mapping $\Phi_\pi : \mathbf{W} \times \mathbf{V} \times \mathbf{\Pi} \to \mathbf{\Pi}$ be defined by

$$\begin{aligned}\Phi_\pi(w, v, \pi) &= \{(\lambda\theta_n + (1-\lambda)\omega, \theta_b) \text{ for some} \\ &\quad (\omega, (\theta_n, \theta_b)) \in \Omega(w, \pi) \times \Theta(w, v, \pi)\}.\end{aligned}$$

Finally, we let the mapping Φ on $\mathbf{W} \times \mathbf{V} \times \mathbf{\Pi}$ be defined by

$$\Phi(w, v, \pi) = (\Phi_w(w, v, \pi), \Phi_v(w, v, \pi), \Phi_\pi(w, v, \pi)).$$

Next, we show that Φ maps $\mathbf{W} \times \mathbf{V} \times \mathbf{\Pi}$ into $\mathbf{W} \times \mathbf{V} \times \mathbf{\Pi}$.

Lemma 1 $\Phi_w(w, v, \pi) \in \mathbf{W}$ *and* $\Phi_\pi(w, v, \pi) \in \mathbf{\Pi}$

Proof. The proof is standard and is omitted. For reference, see the proof of [11, Proposition 1]. ■

Lemma 2 $\Phi_v(w, v, \pi) \in \mathbf{V}$.

Proof. Let $(w, v, \pi) \in \mathbf{W} \times \mathbf{V} \times \mathbf{\Pi}$. Let $h(\cdot) \equiv v(0, \cdot, 0, 0)$. Consider a banker with y who takes w and v as the next date's value functions and π as this date's distribution. The afternoon problem of the banker is described in (7) and (8). It follows from (31) that $y'_m = 0$ and from (32) and the first inequality in (2) that $y'_g = 0$. Therefore, the banker must choose $y'_b = y_m + y_b + d_a - x - y_g$. Then using (9), we have

$$\begin{aligned}g(d_a, d_l, y, x, v) &= ry_b - \rho(x + y_g) - \tau(y_g + y_n + d_l) - \iota \\ &\quad \beta v(0, y_m + y_b + d_a - x - y_g, 0, y_n + d_l - x)\end{aligned}$$

Next, applying (33) with $s = -y'_n = -(y_n + d_l - x)$ and letting $\tilde{y} = y_m + y_b - y_g - y_n$, we have

$$\begin{aligned}g(d_a, d_l, y, x, v) &= ry_b - \rho(x + y_g) - \tau(y_g + y_n + d_l) - \iota \\ &\quad +\beta h(\tilde{y} + d_a - d_l) + \beta\eta(y_n + d_l - x).\end{aligned}$$

It follows that

$$E_x g(d_a, d_l, y, x, v) = \beta h(\tilde{y} + d_a - d_l) + (\beta\eta - \tau)d_l + A(y), \tag{35}$$

where $A(y) = ry_b - (\rho + \tau)y_g + (\beta\eta - \tau)y_n - (\beta\eta + \rho)E_x x$.

Now, we are ready to consider the maximization in (14)-(16). By concavity of w and h, the maximizations over lotteries in (14)-(16) are equivalent to deterministic maximizations with w and h replaced by their extensions to $[0, Z]$ via linear interpolation. Therefore, we can let $(\hat{q}, \hat{d}_a, \hat{d}_l)$ denote a maximizer. If $\hat{d}_a < z$ (the pre-trade wealth of the nonbanker which is an upper bound on d_a), then $(\hat{q}, z, \hat{d}_l + z - \hat{d}_a)$ is also feasible. By the second inequality in (2), $\beta\eta - \tau > 0$. Then by (35), $(\hat{q}, z, \hat{d}_l + z - \hat{d}_a)$ produces a higher value of f_i. Therefore, a necessary condition for a maximum is $d_a = z$. It follows that the lower bound on d_a in (10) can be omitted. Then the constraint sets in (14)-(16) depend only on $\tilde{y}$, the wealth implied by y. Moreover, because $d_a = z$ holds for every banker, it follows that $E_x x = \lambda y_n$. Therefore, we can write

$$f_i(z, y, w, v) = F_i(z, \tilde{y}, w, v) + A^*(y), \tag{36}$$

where $A^*(y) = A(y)$ but with $E_x x = \lambda y_n$. That is,

$$A^*(y) = ry_b - (\rho + \tau)y_g + (\beta\eta - \tau)y_n - (\beta\eta + \rho)\lambda y_n. \tag{37}$$

It follows from (34) and (36) that for $y, y' \in \mathbf{Y}$ with the same implied wealth,

$$\Phi_v(w, v, \pi)(y') - \Phi_v(w, v, \pi)(y) = A^*(y') - A^*(y). \tag{38}$$

Then it follows from (38) and (37) that $\Phi_v(w, v, \pi)$ satisfies (31)-(33). (For (33), let $y' = (y_m, y_b + s, y_g, y_n + s)$, then $A^*(y') - A^*(y) = [r + (\beta\eta - \tau) - (\beta\eta + \rho)\lambda]s = \eta s$.) Finally, it is easy to verify that for all z, $F_i(z, z' + 1, w, v) > F_i(z, z', w, v)$ if $z' < Z$ and $2F_i(z, z', w, v) \geq F_i(z, z' + 1, w, v) + F_i(z, z' - 1, w, v)$ if $0 < z' < Z$. It follows that $\Phi_v(w, v, \pi)$ is nondecreasing and concave in its second argument. ■

Now, we equip $\mathbf{V}$ with the topology of pointwise convergence (on $\mathbf{Y}$). We have the following results.

Lemma 3 $\mathbf{V}$ *is compact.*

Proof. Notice that $v \in \mathbf{V}$ is completely determined by $v(0, \cdot, 0, 0)$. Hence, $\mathbf{V}$ is isomorphic to the set of functions $h : \mathbf{Z} \to \mathbb{R}$ with $h(\cdot) = v(0, \cdot, 0, 0)$ for some $v \in \mathbf{V}$. That set is compact. ■

Lemma 4 Φ *is u.h.c., compact valued, and convex valued.*

Proof. The result follows from the Theorem of Maximum and the convexification by lotteries. ■

We complete the proof of Proposition 1 by way of the next lemma. Recall that we have the following assumptions on parameters: $u'(0)$, $\bar{z}$, and $Z/\bar{z}$ are sufficiently large, and r, ρ, and τ are sufficiently small.

Lemma 5 *There exists $(w, v, \pi) \in \Phi(w, v, \pi)$ in which w is strictly increasing and strictly concave, $h(\cdot) \equiv v(0, \cdot, 0, 0)$ is strictly concave and satisfies $h(z+1) - h(z) \geq \frac{r}{1-\beta}$, and π satisfies $0 < (1-\alpha)\sum_z \pi_n(z)z < \bar{z}$.*

Proof. First, we sketch the proof that there is a fixed point of Φ that satisfies the claim that w is bounded away from 0 at all positive money holdings. (Note that $w(0) = 0$.) Here, we apply arguments used in Zhu [14, Proposition 1]. Indeed, because nonbankers do not profit from their meetings with bankers, aside from somewhat different meeting-rate parameters and a somewhat different law of motion for the evolution of their wealth (the law of motion of their wealth is affected by their meetings with bankers), the nonbankers in this model are exactly like the agents in [14]. For the properties of w, the only aspect of the distribution needed is that a sufficient mass of nonbankers are poor enough relative to $\bar{z}$. Hence, as now explained, the arguments in [14, Proposition 1] are easily adapted.

First, we define $\Phi_n(w, v, \pi) = \Phi(w + 1/n, v, \pi)$, where n is a natural number. Next, from Fan's Fixed Point Theorem (a generalized version of Kakutani's) and Lemmas 3 and 4, we conclude that Φ_n has a fixed point. Next, we prove, exactly as in [14, Lemma 3], that for $(w, v, \pi) \in \Phi_n(w, v, \pi)$, $w(8\bar{z}) \geq D/\beta - 1/n$, where D is the unique solution of $u'(D) = [2/(R\beta)]^2$ with $R = [N(1-\lambda\beta) - (1-\lambda)(N-1)\beta - (1-\lambda)]^{-1}$. In this step, we need the condition that $\sum_{z \leq 4\bar{z}} \pi_n(z) \geq 1/2$. Because $\alpha < 1/2$ and $(1-\alpha)\sum_z \pi_n(z)z \leq \bar{z}$, the condition is satisfied. Next, for a sequence of (w_n, v_n, π_n) with $(w_n, v_n, \pi_n) \in \Phi_n(w_n, v_n, \pi_n)$, there exists a limit point, denoted (w, v, π). It follows that $(w, v, \pi) \in \Phi(w, v, \pi)$ with $w(8\bar{z}) \geq D/\beta$.

The property that $h(z+1) - h(z) \geq \frac{r}{1-\beta}$ is easily verified (one possible use of an additional unit of banker wealth is to acquire a bond).

Next, we consider the claim that both bankers and nonbankers have positive average wealth. First, assume by contradiction that $\sum_z \pi_n(z)z = 0$. Then the measure of bankers with positive wealth is positive and the measure of nonbankers with positive wealth is zero. Then with positive probability, a banker with positive wealth can issue 1 unit of his own notes to a nonbanker with zero wealth. The payoff is $u[\beta w(1)] - \beta\lambda\rho$. As shown above, $w(8\bar{z}) \geq D/\beta$. This and concavity of w imply that the payoff is positive for sufficiently small ρ. (As noted above, small ρ should be interpreted as sufficient asset divisibility.) Hence, such a banker issues a positive amount of notes, which contradicts $\sum_z \pi_n(z)z = 0$.

Next, assume by contradiction that $\sum_z \pi_n(z)z = \bar{z}/(1-\alpha)$, so that bankers have no wealth. Then, there is no trade of goods between bankers and nonbankers. In this case, even the law of motion for nonbanker wealth is unaffected by their meetings with bankers. Then the nonbankers are exactly like the agents in [14], and the full-support result in [14, Lemma 9] holds and implies that π_n has full support. Because w is bounded above, for sufficiently large z, $w(z+1)-w(z) < \frac{r}{1-\beta}$. As noted above, for all z, $h(z+1)-h(z) \geq \frac{r}{1-\beta}$. Hence, all bankers are willing to produce for a nonbanker with sufficiently high z to get at least 1 unit of wealth. But this contradicts $\sum_z \pi_b(z)z = 0$.

Finally, the fact that w is strictly increasing is easily confirmed. (One may apply the corresponding simple argument in [14, Lemma 5]). And strict concavity of w and h can also be easily confirmed. ■

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