Option Pricing under a Mixed-Exponential Jump Diffusion Model

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December 2008; this version, May 2011

Abstract

This paper aims at extending the analytical tractability of the Black-Scholes model to alternative models with arbitrary jump size distributions. More precisely, we propose a jump diffusion model for asset prices whose jump sizes have a mixed-exponential distribution, which is a weighted average of exponential distributions but with possibly negative weights. The new model extends existing models, such as hyper-exponential and double-exponential jump diffusion models, as the mixed-exponential distribution can approximate any distribution as closely as possible, including the normal distribution and various heavy-tailed distributions. The mixed-exponential jump diffusion model can lead to analytical solutions for Laplace transforms of prices and sensitivity parameters for path-dependent options such as lookback and barrier options. The Laplace transforms can be inverted via the Euler inversion algorithm. Numerical experiments indicate that the formulae are easy to implement and accurate. The analytical solutions are made possible mainly because we solve a high-order integro-differential equation explicitly. A calibration example for SPY options shows that the model can provide a reasonable fit even for options with very short maturity, such as one day.

Key words: jump diffusion, mixed-exponential distributions, lookback options, barrier options, Merton’s normal jump diffusion model, first passage times

1 Introduction

1.1 Background

It is well known that empirically asset return distributions have heavier left and right tails than those of the normal distributions, as suggested in the classical Black-Scholes model. Jump diffusion models are among the most popular alternative models proposed to address this issue, and they are especially useful to price options with short maturities. For the background of alternative models, see, e.g., Hull [30]. This paper aims at further extending the analytical
tractability of the Black-Scholes model to jump diffusion models with arbitrary jump size distributions. Indeed, we propose a jump diffusion model for asset prices whose jump sizes have a mixed-exponential distribution, which is a weighted average of exponential distributions but with possibly negative weights. The mixed-exponential distribution can approximate any distribution arbitrarily closely, including any discrete distribution, the normal distribution, and various heavy-tailed distributions such as Gamma, Weibull and Pareto distributions. We show that the mixed-exponential jump diffusion model can lead to analytical solutions for Laplace transforms of prices and sensitivity measures (e.g., deltas) for path-dependent options, such as continuously-monitored lookback and barrier options. These analytical solutions are made possible primarily because we solve a high-order integro-differential equation explicitly related to the first passage time problem.

The motivation of the paper is two-fold. First, a key question for jump diffusion models is what jump size distributions will be used. The question is closely related to how heavy the tails of asset return distributions are. Although we know that asset return distributions have heavier tails than the normal distribution, it is not clear at all how heavy the tails may be. For example, empirically it may be difficult to identify how heavy the tails are based on 5,000 (about 20 years) daily observations; see Heyde and Kou [28]. Accordingly, we want the jump size distribution to be general enough to approximate any distribution, including various exponential- and power-tail distributions.

Second, analytical tractability is one of the challenges for alternative models to the Black-Scholes model. More precisely, although many alternative models can lead to analytical solutions for European call and put options, unlike the Black-Scholes model, it is difficult to do so for path-dependent options such as lookback and barrier options. Even numerical methods for these derivatives are not easy. For example, the convergence rates of binomial trees and Monte Carlo simulation for path-dependent options are typically much slower than those for call and put options; for a survey, see, e.g., Boyle et al. [12].

Therefore, it is desirable to have a class of jump diffusion models that allow jump size distributions that can approximate any distribution while remaining tractable enough to allow analytical solutions for path-dependent options. We shall show that this is possible if we consider a mixed-exponential jump diffusion model (MEM).

1.2 Comparison with the Existing Literature

Two well-known jump diffusion models are Merton’s model [40] and the double-exponential jump diffusion model (see Kou [33]), in which the jump size distributions are normal and double exponential, respectively. One advantage of the double-exponential jump diffusion model is that
it can lead to analytical tractability for path-dependent options, including lookback, barrier, Asian, and occupation-time related options; see, e.g., Kou and Wang [36], Cai and Kou [15] and Cai et al. [14]. More general models, including the phase-type jump diffusion model (PHM) and the hyper-exponential jump diffusion model (HEM), were also proposed; see, e.g., Asmussen et al. [2], Boyarchenko [8], Boyarchenko and Boyarchenko [9], Boyarchenko and Levendorskiı̆ [11], Cai et al. [13], Carr and Crosby [17], Crosby et al. [22], Jeannin and Pistorius [31], Lipton [38]. Here is a comparison between our MEM model and the existing HEM and PHM models.

(1) The HEM specifies the jump size distribution as a weighted average of exponential distributions, and the weights can only be nonnegative. Therefore, the HEM is a special case of our MEM model, because our weights can be negative. Compared with the HEM which can only approximate jump diffusion models with completely monotone jump size distributions (see Appendix B for more details), our MEM can approximate jump diffusion models with *any* jump size distribution, because the mixed-exponential distribution is dense with respect to the class of all the distributions in the sense of weak convergence (see Botta and Harris [7]). In particular, the MEM may be used to approximate Merton’s model, which cannot be approximated by the HEM as the normal distribution is not completely monotone. In Section 6, an example will be provided to demonstrate that this approximation can lead to accurate prices and deltas for lookback and barrier options under the Merton’s model.

(2) The PHM, in which the jump sizes have a phase-type distribution, can also approximate jump diffusion models with any jump size distribution (see [7]). One issue worth mentioning is that the class of the MEM and that of the PHM do not contain each other. Moreover, one advantage of the MEM might be the representation of the mixed-exponential distribution is *unique*, whereas that of the phase-type distribution is *not unique* (see [7]); i.e., for phase-type distributions different sets of parameters may lead to the same cumulative distribution function (cdf). The uniqueness is desirable for statistical procedures such as parameter estimation.

In terms of the related literature on pricing path-dependent options, Feng and Linetsky [26] and Feng et al. [27] showed how to price path-dependent options numerically, via extrapolation and variational methods, for jump diffusion models with general jump size distributions. Davydov and Linetsky ([23], [24]) provided analytical pricing formulae for lookback and barrier options under the CEV model. For option pricing under exponential Lévy models, see Carr et al. [18], Cont and Tankov [21], and Kijima [32]. The emphasis of the current paper is on explicit calculations for a particular exponential Lévy model, which are different from these results.

We point out that none of the exponential Lévy models can capture both short- and long-term behaviors of market options. In fact, the jump diffusion models are useful especially
for short maturity options. In general, to get an excellent fit across all strikes and all option maturities, spatial inhomogeneity and/or stochastic volatilities may be used; see, e.g., Bates [6], Bakshi et al. [4] and Carr et al. [18]. Therefore, the formulae given in this paper are only meant to be a first step to price options analytically under more general models with jumps. However, the analytical formulae presented here may be useful for short-term options, and can also provide a useful benchmark for more complicated models, for which one perhaps has to resort to simulation or other numerical procedures.

The paper is organized as follows. Section 2 gives the basic setting of the MEM and provides motivation and intuition of our results. First passage times of the mixed-exponential jump diffusion process are studied in Section 3. Section 4 discusses pricing of European options and provides an example of calibration to a set of data of European options. Laplace transforms of prices and deltas for lookback and barrier options are given in Section 5, where numerical results are also provided. In Section 6, a numerical example is given to illustrate an approximation to Merton’s model by the MEM, especially in terms of lookback and barrier options. Section 7 concludes the paper. The proofs are deferred to the appendices or the electronic companion.

2 The Mixed-Exponential Jump Diffusion Model

Under the mixed-exponential jump diffusion model (MEM), the dynamics of the asset price $S_t$ under a risk-neutral measure $^{1}$ $P$ to be used for option pricing is given by

$$
\frac{dS_t}{S_{t-}} = rdt + \sigma dW_t + d \left( \sum_{i=1}^{N_t} (V_i - 1) \right),
$$

where $r$ is the risk-free interest rate, $\sigma$ the volatility, $\{N_t : t \geq 0\}$ a Poisson process with rate $\lambda$, $\{W_t : t \geq 0\}$ a standard Brownian motion, and $\{Y_i := \log(V_i) : i = 1, 2, \ldots\}$ a sequence of independent identically distributed (i.i.d.) mixed-exponential random variables with the probability density function (pdf) $f_Y(x)$. In this model all sources of randomness, $N_t$, $W_t$, and $Y_i$'s, are assumed to be independent.

More precisely, the pdf $f_Y(x)$ is given by

$$
f_Y(x) = p_u \sum_{i=1}^{m} p_i \eta_i e^{-\eta_i x} I_{x \geq 0} + q_d \sum_{j=1}^{n} q_j \theta_j e^{\theta_j x} I_{x < 0},
$$

Due to the jumps in the model, there are many risk-neutral probability measures. It can be shown (see, e.g., Kou [33]) that by using the rational expectations argument with a HARA type utility function for the representative agent, one can choose a particular risk-neutral measure $P$ so that the equilibrium price of an option is given by the expectation under this risk neutral measure of the discounted option payoff. The measure $P$ is called risk-neutral because $E(e^{-rt}S_t) = S_0$. 

1
where \( p_u \geq 0, \ q_d = 1 - p_u \geq 0, \)
\[
p_i \in (-\infty, \infty) \quad \text{for all } i = 1, \ldots, m, \quad \sum_{i=1}^{m} p_i = 1,
\]
\[
q_j \in (-\infty, \infty) \quad \text{for all } j = 1, \ldots, n, \quad \sum_{j=1}^{n} q_j = 1,
\]
\[
\eta_i > 1 \quad \text{for all } i = 1, \ldots, m \quad \text{and} \quad \theta_j > 0 \quad \text{for all } j = 1, \ldots, n.
\]

Since \( p_i \) and \( q_j \) can be negative, the parameters should satisfy some conditions to guarantee that \( f_Y(x) \) is always nonnegative and is a probability density function. A necessary condition for \( f_Y(x) \) to be a probability density function is \( p_1 > 0, \ q_1 > 0, \sum_{i=1}^{m} p_i \eta_i \geq 0, \) and \( \sum_{j=1}^{n} q_j \theta_j \geq 0. \)

A simple sufficient condition is \( \sum_{i=1}^{k} p_i \eta_i \geq 0 \) for all \( k = 1, \ldots, m \) and \( \sum_{j=1}^{l} q_j \theta_j \geq 0 \) for all \( l = 1, \ldots, n. \) For alternative conditions, see Bartholomew [5]. A special case of the mixed-exponential distribution is the hyper-exponential distribution, where all the \( p_i \) and \( q_j \) must be nonnegative.

In addition, the condition \( \eta_i > 1, \) for all \( i = 1, \ldots, m, \) is imposed above to ensure that the stock price \( S_t \) has a finite expectation. By solving the stochastic differential equation (1), we obtain that under the MEM the return process \( X_t := \log(S_t/S_0) \) is given by
\[
X_t = \mu t + \sigma W_t + \sum_{i=1}^{N_t} Y_i, \quad X_0 = 0,
\]
where \( \mu = r - \frac{\sigma^2}{2} - \lambda \zeta \) and \( \zeta := E[e^{Y_1}] - 1 = p_u \sum_{i=1}^{m} \frac{p_i \eta_i}{\eta_i - 1} + q_d \sum_{j=1}^{n} \frac{q_j \theta_j}{\theta_j + 1} - 1. \)

Simple algebra yields that the moment generating function of \( X_t \) is
\[
E[e^{xX_t}] = e^{G(x)t}, \quad \text{for any } t \geq 0 \text{ and } x \in (-\theta_1, \eta_1),
\]
where \( G(x) \), called the exponent of the Lévy process \( X_t \), is defined as
\[
G(x) = \frac{\sigma^2}{2} x^2 + \mu x + \lambda \left( p_u \sum_{i=1}^{m} \frac{p_i \eta_i}{\eta_i - x} + q_d \sum_{j=1}^{n} \frac{q_j \theta_j}{\theta_j + x} - 1 \right).
\]

For more information about exponents of this type, we refer to Hirschman and Widder [29].

Besides, the infinitesimal generator of \( \{X_t\} \) is given by
\[
(Lu)(x) = \frac{\sigma^2}{2} u''(x) + \mu u'(x) + \lambda \int_{-\infty}^{+\infty} [u(x + y) - u(x)] f_Y(y) dy,
\]
where \( u(x) \) is any twice continuously differentiable function and \( f_Y(\cdot) \) is given by (2).
The difficulty in distinguishing tail behaviors (see Heyde and Kou [28]) motivates us to consider the MEM, whose jump size distribution is general enough to approximate any jump size distribution, no matter which ones we prefer. In fact, the mixed-exponential (ME) distribution can approximate any distribution in the sense of weak convergence (see Botta and Harris [7]).

We shall provide several examples of approximating some heavy-tailed distributions numerically with the ME, including (a) Gamma (1.2, 0.5), i.e., the Gamma distribution with shape parameter 1.2 and scale parameter 0.5; (b) Gamma (0.8, 0.85); (c) Pareto (2, 25), i.e., the Pareto distribution with shape parameter 2 and scale parameter 25; and (d) Weibull (0.025, 0.5), i.e., the Weibull distribution with scale parameter 0.025 and shape parameter 0.5. Note that (b)-(d) are completely monotone, but (a) is not and hence cannot be approximated by the hyper-exponential distributions. Besides, although theoretically phase-type distributions can approximate (a)-(d), the numerical fitting might not be easy because the representation of a phase-type distribution is not unique (see Botta and Harris [7]).

In our examples, we first fix the number \( m \) of components of the mixed-exponential distribution, whose cdf is denoted by \( \text{MExp}_m(x) \). Then an approximation to the target cdf \( H(x) \) by a mixture of \( m \) exponential distributions can be found by minimizing \( \sum_{i=1}^{N} (\text{MExp}_m(x_i) - H(x_i))^2 \), where \( x_1, \cdots, x_N \) are grid points on some interval. Figure 1 suggests that it seems possible to use a mixture of two, three, three and five exponential distributions to fit Gamma (1.2, 0.5), Gamma (0.8, 0.85), Pareto (2, 25) and Weibull (0.025, 0.5), respectively.

3 First Passage Times

To price lookback and barrier options, it is pivotal to study the first passage times \( \tau_b \) that the process crosses a flat boundary with a level \( b \), where

\[
\tau_b := \inf\{t \geq 0 : X_t \geq b\}, \quad b > 0,
\]

and the infimum of an empty set is defined as \( +\infty \) and \( X_{\tau_b} := \limsup_{t \to +\infty} X_t \) on the set \( \{\tau_b = +\infty\} \). When a jump diffusion process crosses the boundary, sometimes it hits the boundary exactly and sometimes it incurs an “overshoot”, \( X_{\tau_b} - b \), over the boundary\(^2\). The overshoot presents several problems if one wants to compute the distribution of the first passage

\(^2\)If the jump size distribution is one-sided, one can solve the overshoot problems by either using renewal equations or fluctuation identities for Lévy processes; see, e.g., Avram, Chan and Usabel [3] and Rogers [42]. However, for two-sided jumps, because of the ladder-variable problems, generally speaking the renewal equations are not available and the fluctuation identities for arbitrary distributions become too complicated for explicit computation; see, e.g., the discussion in Siegmund [44] and Rogers [42]. See also Boyarchenko and Levendorskiï [10] and Kyprianou and Pistorius [37] for some representations related to the overshoot problems for general Lévy processes.
Figure 1: Approximate heavy-tailed distributions (Gamma, Pareto, and Weibull) with mixed-exponential distributions. This figure suggests that it seems possible to use a mixture of two, three, three and five exponential distributions to fit Gamma (1.2, 0.5), Gamma (0.8,0.85), Pareto (2.25) and Weibull (0.025, 0.5), respectively. Note that Gamma (1.2, 0.5) is not completely monotone and hence cannot be approximated by the hyper-exponential distribution. For Gamma (1.2, 0.5), a plotted approximation is \(1 + e^{-1.4014x} + e^{-7.5316x}\). For Gamma (0.8, 0.85), a plotted approximation is \(0.8435(1-e^{-1.2937x}) + 0.1305(1-e^{-5.4092x}) + 0.0260(1-e^{-70.0207x})\). For Pareto (2, 25), a plotted approximation is \(0.0841(1-e^{-5.7140x}) + 0.5165(1-e^{-27.5604x}) + 0.3994(1-e^{-86.7169x})\). For Weibull (0.025, 0.5), a plotted approximation is \(0.1411(1-e^{-5.1891x}) + 0.1604(1-e^{-14.5982x}) + 0.2519(1-e^{-29.4403x}) + 0.2734(1-e^{-135.0813x}) + 0.1732(1-e^{-2000x})\).

time analytically. First, one needs the exact distribution of the overshoot, \(X_t-b\); particularly, \(P(X_t-b=0)\) and \(P(X_t-b>x)\) for \(x>0\). Second, one needs to know the dependence structure between the overshoot, \(X_t-b\), and the first passage time \(\tau_b\).

These difficulties can be resolved if one can solve the following ordinary integro-differential equation (OIDE) explicitly.

\[
\begin{cases}
(Lu)(x) - \alpha u(x) = \frac{\alpha^2}{2} u''(x) + \mu u'(x) - (\lambda + \alpha) u(x) + \lambda \int_{-\infty}^{+\infty} u(x+y) f_Y(y) dy = 0, & \text{if } x < x_0 \\
u(x) = g(x), & \text{if } x \geq x_0,
\end{cases}
\]

(6)

where \(\alpha > 0\), \(L\) is the infinitesimal generator of \(\{X_t\}\) given by (4), and \(g(x)\) is a given function.

Note that the main challenge is that although this OIDE exists only when \(x < x_0\), it does involve the information of the function \(u(x)\) for \(x \geq x_0\) because the integral inside the generator \(L\)
depends on the values of $u$ on both regions. To emphasize the above particularity of the OIDE, we call it a \textit{forced OIDE}, meaning that the OIDE has a forcing term defined by $g(x)$. In this section, this OIDE will be solved explicitly, leading to an analytical solution of the joint distribution of the first passage time $\tau_b$ and $X_t$. Intuitively, the solution is available analytically because the exponential function has some very nice properties, such as the product of exponential functions is still an exponential function, and the derivatives of exponential functions are still exponential functions.

It is worth noting that our argument requires neither the Weiner-Hopf factorization nor more general theory about Markov processes. We prove the main results by solving the OIDE (6) explicitly and by using a martingale method. More specifically, we will achieve the objective in four steps: (i) Show that $G(x) = \alpha$ has only real roots for any sufficiently large $\alpha > 0$. (ii) Use the roots to solve the OIDE (6) explicitly by transforming the OIDE into a high-order homogeneous linear ordinary differential equation (ODE). Some indications of possible reduction of an OIDE to a high-order ODE are also given in Mayo [39] and Carr et al. [19]. (iii) Derive $E_x[e^{-\alpha \tau_b + \theta X_t}]$ via a martingale method based on the solution of the OIDE, where the superscript $x$ means $X_0 = x$. (iv) Obtain the double Laplace transform of the joint distribution of $\tau_b$ and $X_t$ using the result in Step (iii).

(i) Roots of the equation $G(x) = \alpha$.

\textbf{Theorem 3.1} For sufficiently large $\alpha > 0$, the equation $G(x) = \alpha$ has $(m + n + 2)$ roots that are all real and are distinct. Specifically, we have $(m + 1)$ positive roots, $\beta_1, \alpha, \ldots, \beta_{m+1}, \alpha$, and $(n + 1)$ negative roots, $\gamma_1, \alpha, \ldots, \gamma_{n+1}, \alpha$, as follows.

$$-\infty < \gamma_{n+1}, \alpha < \cdots < \gamma_2, \alpha < \gamma_1, \alpha < 0 < \beta_1, \alpha < \beta_2, \alpha < \cdots < \beta_{m+1}, \alpha < +\infty. \quad (7)$$

The proof is given in Section A of the electronic companion. Figure 2 illustrates the function $G(x)$, from which we can see how the roots behave for sufficiently large $\alpha > 0$.

(ii) Solving the OIDE (6) explicitly.

A technical contribution of the current paper is that we solve explicitly the forced OIDE (6), by transforming it into a homogeneous linear ODE.

\textbf{Theorem 3.2} Assume that $\alpha > 0$ is sufficiently large such that $G(x) = \alpha$ has $(m + n + 2)$ real roots, satisfying (7). Then any solution $u(x)$ of OIDE (6) is also a solution of an $(m + n + 2)$ order homogeneous linear ODE with constant coefficients, whose characteristic equation is given by $(G(x) - \alpha) \prod_{i=1}^m (x - \eta_i) \prod_{j=1}^n (x + \theta_j) = 0$. Thus, any solution of OIDE (6) is of the form

$$u(x) = \sum_{i=1}^{m+1} c_i e^{\beta_i, \alpha x} + \sum_{j=1}^{n+1} d_j e^{\gamma_j, \alpha x}, \quad (8)$$

8
where $c_1, c_2, \cdots, c_{m+1}, d_1, d_2, \cdots, d_{n+1}$ are undetermined constants.

Proof. See Section B in the electronic companion. □

(iii) Joint distribution of the first passage time $\tau_b$ and the overshoot $X_{\tau_b} - b$.

**Theorem 3.3** For any sufficiently large $\alpha > 0$, $\theta < \eta_1$ and $x, b \in \mathbb{R}$, we have

$$E^x[e^{-\alpha \tau_b + \theta X_{\tau_b}}] = \begin{cases} e^{\theta x} & \text{if } x \geq b \\ \sum_{l=1}^{m+1} w_l e^{\beta_l \alpha x} & \text{if } x < b, \end{cases}$$

where $x = X_0$ and $\beta_{1,\alpha}, \cdots, \beta_{m+1,\alpha}$ are the $m+1$ positive roots of the equation $G(x) = \alpha$ such that $0 < \beta_{1,\alpha} < \beta_{2,\alpha} < \cdots < \beta_{m+1,\alpha}$. Here $w := (w_1, w_2, \cdots, w_{m+1})'$ is a vector uniquely determined by the following linear system

$$ABw = J,$$

where $A$ is an $(m+1) \times (m+1)$ nonsingular matrix

$$A = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \eta_1 \eta_2 & \eta_1 \eta_2 & \cdots & \eta_1 \eta_{m+1} \\ \eta_2 \eta_1 - \beta_{1,\alpha} & \eta_2 \eta_1 - \beta_{2,\alpha} & \cdots & \eta_2 \eta_{m+1} - \beta_{m+1,\alpha} \\ \vdots & \vdots & \ddots & \vdots \\ \eta_m \eta_1 & \eta_m \eta_2 & \cdots & \eta_m \eta_{m+1} \end{pmatrix}.$$
$B$ is an $(m+1) \times (m+1)$ diagonal matrix and $J$ is an $(m+1)$-dimensional vector

$$B = \text{Diag}\{e^{\beta_1a}, e^{\beta_2a}, \ldots, e^{\beta_{m+1}a}\}, \quad J = e^{\theta b} \left(1, \frac{\eta_1}{\eta_1 - \theta}, \frac{\eta_2}{\eta_2 - \theta}, \ldots, \frac{\eta_m}{\eta_m - \theta}\right)'.$$

(12)

In particular, with $\theta = 0$, we have for sufficiently large $\alpha > 0$

$$E_x[e^{-\alpha \tau}] = \begin{cases} 1 & \text{if } x \geq b \\ \sum_{l=1}^{m+1} c_le^{\beta_l a}x & \text{if } x < b. \end{cases}$$

(13)

Here $c := (c_1, c_2, \ldots, c_{m+1})'$ is a positive vector uniquely determined by the linear system

$$ABc = 1,$$

(14)

where $1 = (1, 1, \ldots, 1)'$.

**Proof.** See Section C in the electronic companion. □

(iv) Joint distribution of $\tau_b$ and $X_t$.

Without loss of generality, we assume $X_0 = 0$. The joint distribution of $\tau_b$ and $X_t$, i.e.,

$$P^0(X_t \geq a, \tau_b \leq t) = P^0(X_t \geq -\hat{a}, \tau_b \leq t),$$

(15)

for some fixed numbers $a \equiv -\hat{a} \leq b$ and $b > 0$, has a variety of applications, including pricing barrier options.

**Theorem 3.4** Assume $X_0 = 0$. Denote by $L(\alpha, \theta)$ the double Laplace transform of $P^0(X_t \geq -\hat{a}, \tau_b \leq t)$ w.r.t. $t$ and $\hat{a}$, respectively, i.e.,

$$L(\alpha, \theta) = \int_0^{+\infty} \int_{-\infty}^{+\infty} e^{-\theta \hat{a} - \alpha t} P^0(X_t \geq -\hat{a}, \tau_b \leq t)d\hat{a}dt.$$

Then for any $\theta \in (0, \eta_1)$ and sufficiently large $\alpha > \max(G(\theta), 0)$, we have

$$L(\alpha, \theta) = \sum_{l=1}^{m+1} \frac{\hat{d}_l e^{-\beta_l a b}}{\theta(\alpha - G(\theta))},$$

(16)

where $\beta_1, \alpha, \ldots, \beta_{m+1}, \alpha$ are the $m+1$ positive roots of the equation $G(x) = \alpha$ such that $0 < \beta_1, \alpha < \beta_2, \alpha < \cdots < \beta_{m+1}, \alpha$. Here $\hat{d} := (\hat{d}_1, \hat{d}_2, \cdots, \hat{d}_{m+1})'$ solves the linear system $A \hat{d} = J$, where $A$ and $J$ are the same as in (11) and (12).

**Proof.** See Section D in the electronic companion. □
4 Pricing European Options under the MEM and a Calibration Example

4.1 Pricing European Options under the MEM

Under the risk neutral measure $P$, the value of a European call option with a fixed strike $K$ and maturity $T$ is given by $e^{-rT}E[(S_T - K)^+]$. To apply the two-sided Euler inversion algorithm proposed by Petrella [41], we introduce a scaling factor $X > K$, which ensures that the Euler inversion algorithm converges quickly. The call option value can then be expressed as

$$C_T(k_c) = e^{-rT}X \cdot E\left[\frac{S_T}{X} - e^{-k_c}\right],$$

where $k_c = \log(X/K)$. The same proof as that for Lemma 1 in [34], which is based on Carr and Madan [20], leads to the Laplace transform of $C_T(k_c)$ w.r.t. $k_c$.

$$\hat{C}(\psi) := \int_{-\infty}^{+\infty} e^{-\psi k_c} C_T(k_c) dk_c = e^{-rT}S_0^\psi e^{G(\psi+1)T/\psi(\psi+1)}X^\psi, \quad \text{for any } \psi \in (0, \eta_1 - 1), \quad (17)$$

where $G(\cdot)$ is given by (3). As an immediate result, we can obtain the closed-form Laplace transform of the European option delta $\Delta_T(k_c) := \frac{\partial C_T(k_c)}{\partial S_0}$ w.r.t. $k_c$.

$$\hat{\Delta}(\psi) := \int_{-\infty}^{+\infty} e^{-\psi k_c} \Delta_T(k_c) dk_c = e^{-rT}S_0^\psi e^{G(\psi+1)T/\psi(\psi+1)}X^\psi, \quad \text{for any } \psi \in (0, \eta_1 - 1), \quad (18)$$

where the interchange of derivatives and integrals can be justified by Theorem A. 12 on page 203-204 in Schiff [43].

Inverting $\hat{C}(\psi)$ and $\hat{\Delta}(\psi)$ via the two-sided Euler inversion method\(^3\) yields numerical results of European call option prices and deltas under the MEM, which are provided in Table 1. It can be seen that all of our numerical results (denoted by EI values) stay within the 95% confidence intervals of the associated Monte Carlo simulation estimates (denoted by MC values), which are obtained by using $S_T$ as a control variate to reduce variances. All the computations in this paper are conducted on a Laptop with a Duo 2.50GHz CPU.

Since a European call option is the same as a degenerated up-and-in call barrier option with barrier $H = S_0$, the column “BA value” in Table 1 also reports the results using the degenerated barrier options by numerically inverting the double Laplace transforms (22) and

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\(^3\)Note that Petrella’s algorithm is faster and more stable than the original Euler inversion (see [1]) due to the introduction of a scaling factor. In the implementation of the one-sided, one-dimensional (two-sided, two dimensional) Euler inversion method, some parameters $n$ and $A$ ($n_1, n_2, A_1$ and $A_2$) are involved. The parameter $n$ ($n_1$ and $n_2$) is (are) used in the Euler transformation to accelerate the computation of some alternating series in the inversion formula, while $A$ ($A_1$ and $A_2$) is (are) used to control the discretization errors. For more information about these parameters and the inversion formulae, see Petrella [41], Abate and Whitt [1], and Cai et al. [16].
We can see that the numerical European option prices (and deltas) obtained in this way agree with those by inverting \( \hat{C}(\psi) \) in (17) (and \( \hat{\Delta}(\psi) \) in (18)) up to five decimal points.

### Prices of European call options under the MEM

<table>
<thead>
<tr>
<th>( \eta_1 )</th>
<th>( \lambda )</th>
<th>EI value</th>
<th>BA value</th>
<th>MC value</th>
<th>Std Err</th>
<th>EI value</th>
<th>BA value</th>
<th>MC value</th>
<th>Std Err</th>
</tr>
</thead>
<tbody>
<tr>
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<td>10.97472</td>
<td>10.97339</td>
<td>0.01923</td>
<td>14.59752</td>
<td>14.59752</td>
<td>14.59520</td>
<td>0.02776</td>
<td></td>
</tr>
<tr>
<td>20</td>
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<td>11.94485</td>
<td>11.92641</td>
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<td>15.29993</td>
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</tr>
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<td>15.96677</td>
<td>15.99681</td>
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<tr>
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<td>10.82050</td>
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<td>0.01865</td>
<td>14.48475</td>
<td>14.48475</td>
<td>14.46554</td>
<td>0.02740</td>
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</tbody>
</table>

### Deltas of European call options under the MEM

<table>
<thead>
<tr>
<th>( S_0 )</th>
<th>( K )</th>
<th>EI value</th>
<th>BA value</th>
<th>MC value</th>
<th>Std Err</th>
<th>EI value</th>
<th>BA value</th>
<th>MC value</th>
<th>Std Err</th>
</tr>
</thead>
<tbody>
<tr>
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<td>0.74249</td>
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</tr>
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<td>0.63645</td>
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<td>0.62633</td>
<td>0.62633</td>
<td>0.62610</td>
<td>0.00091</td>
<td></td>
</tr>
<tr>
<td>110</td>
<td>0.47150</td>
<td>0.47150</td>
<td>0.47112</td>
<td>0.00100</td>
<td>0.50940</td>
<td>0.50940</td>
<td>0.50895</td>
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</tr>
<tr>
<td>90</td>
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<td>0.81396</td>
<td>0.81368</td>
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<td>0.76198</td>
<td>0.76198</td>
<td>0.76237</td>
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</tr>
<tr>
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<td>0.66848</td>
<td>0.66869</td>
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<td>0.64953</td>
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<td></td>
</tr>
<tr>
<td>110</td>
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<td>0.50648</td>
<td>0.50494</td>
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<td>0.53414</td>
<td>0.53450</td>
<td>0.00097</td>
<td></td>
</tr>
</tbody>
</table>

Table 1: The Euler inversion (EI value or BA value) vs. Monte Carlo simulation (MC value) for calculating the prices and deltas of European options under the MEM. For the “price” part, the default choices for unvarying parameters are \( r = 0.05, \theta_1 = \eta_1, \eta_2 = \theta_2 = 50, p_u = 0.4, q_d = 0.6, p_1 = 1.2, p_2 = -0.2, q_1 = 1.3, q_2 = -0.3, S_0 = 100, K = 100 \) and \( t = 1 \). For the “delta” part, the default choices of unvarying parameters are \( r = 0.05, \lambda = 3, \eta_1 = 30, \eta_2 = 40, \theta_1 = 20, \theta_2 = 30, p_u = 0.4, q_d = 0.6, p_1 = 1.2, p_2 = -0.2, q_1 = 1.3, q_2 = -0.3 \) and \( t = 1 \). Parameters for the Euler inversion method are \( A = 18, n = 30 \) and \( X = 10000 \). The MC values and the associated standard errors (denoted by Std Err) are obtained by simulating 100,000 sample paths and by setting the step size to be 0.00005. Here \( S_T \) is used as a control variate to achieve variance reduction. We can see that all the EI values stay within the 95% confidence intervals of the associated MC values. In addition, the BA values denote the European call option prices or deltas obtained by calculating prices or deltas of up-and-in call barrier options with barrier \( H = S_0 \). It is easily seen that all the BA values agree with the EI values to five decimal points. The CPU times to generate one EI value, one BA value, and one MC value are less than 0.01 seconds, about 5 seconds, and about 50 seconds, respectively.

### 4.2 A Calibration Example

In general model calibration is an important and yet difficult problem that involves various optimization and numerical pricing techniques. In this subsection, we give an example to illustrate the calibration of our MEM to a set of market data. For more comprehensive discussion on the calibration, we refer to Cont and Tankov [21]. In our example, the data set obtained from
Morningstar Inc. consists of the closing prices (i.e. the averages of bid and ask prices) of 47 SPY (S&P 500 ETF stock) European call options on March 29, 2010, with three maturities (1 day, 18 days and 53 days) and various strike prices. Our goal is to calibrate the model to these option prices across different maturities and different strikes using only one set of parameters. The calibration is especially interesting as it is well known that it is difficult to calibrate models to options with very short maturity such as one day.

We shall minimize the objective function \( \sum_{i=1}^{47} \left( \frac{C_i(\pi) - \tilde{C}_i(\pi)}{\text{Vega}(IV_i)} \right)^2 \) over the set of unknown parameters \( \pi = (\sigma, \lambda, \eta_1, \ldots, \eta_m, \theta_1, \ldots, \theta_n, p_1, \ldots, p_m, q_1, \ldots, q_n) \), where \( C_i \) and \( \tilde{C}_i \) represent the calibrated European option price and the market price respectively and \( IV_i \) is the market implied volatility for the \( i \)th option. This objective function for calibration is suggested in Cont and Tankov [21] (see p. 439). For simplicity, we use an MEM with the upward jump distribution being exponential and the downward jump distribution being a mixture of two exponentials (i.e, \( m = 1 \) and \( n = 2 \)). To solve the optimization problem, we first select 100 best starting points from around 20,000 grid points and then search the optimal solution for each of these 100 starting points. The best one is chosen to be our final optimal solution.

Figure 3 shows both observed market implied volatilities and calibrated implied volatilities. It is worth mentioning that although in general it is difficult to fit the implied volatilities for the options with very short maturity such as one day, it seems that our model can produce a close fit even to this sharp volatility skew.

5 Lookback and Barrier Options

5.1 Lookback Options

We shall only consider lookback put options because lookback call options can be obtained by symmetry. Under the risk neutral measure \( P \), the price of a lookback put option with the maturity \( T \) is given by

\[
LP(T) = E \left[ e^{-rT} \left( \max \left\{ M, \max_{0 \leq t \leq T} S_t \right\} - S_T \right) \right] = E \left[ e^{-rT} \max \left\{ M, \max_{0 \leq t \leq T} S_t \right\} \right] - S_0,
\]

where \( M \geq S_0 \) is a fixed constant representing the prescribed maximum at time 0.

**Theorem 5.1** For all sufficiently large \( \alpha > 0 \), the Laplace transforms of the lookback put option price \( LP(T) \) and delta \( \Delta(T) := \frac{\partial LP(T)}{\partial S_0} \) w.r.t. the maturity \( T \) are given by

\[
\int_0^{+\infty} e^{-\alpha T} LP(T) dT = \frac{S_0}{\alpha + r} \sum_{i=1}^{m+1} \frac{d_i}{\beta_i \alpha + r - 1} \left( \frac{S_0}{M} \right)^{\beta_i \alpha + r - 1} + \frac{M}{\alpha + r} - \frac{S_0}{\alpha}, \tag{19}
\]
Figure 3: Calibrated implied volatilities vs. observed market implied volatilities. The initial stock price is 117.32. The risk-free interest rates corresponding to these three maturities, 1 day, 18 days and 53 days, are 0.0011, 0.0011 and 0.0012, respectively. Note that in general it is difficult to fit the implied volatilities for the options with very short maturity one day. However, it seems that our model can produce a close fit even to this sharp volatility skew. The parameters used in the calibrated model are \( \hat{\sigma} = 0.10997 \), \( \hat{\lambda} = 6.19653 \), \( \hat{\eta}_1 = 202 \), \( \hat{\theta}_1 = 45.21588 \), \( \hat{\theta}_2 = 78.40339 \), \( \hat{p}_1 = 0.00077 \), \( \hat{q}_1 = 3.09202 \), and \( \hat{q}_2 = -2.09279 \).

and

\[
\int_0^{+\infty} e^{-\alpha T} \Delta(T) dT = \frac{1}{\alpha + r} \sum_{i=1}^{m+1} d_i \beta_{i,\alpha+r} \left( \frac{S_0}{M} \right)^{\beta_{i+1,\alpha+r} - 1} - \frac{1}{\alpha},
\]

(20)

respectively, where \( \beta_{1,\alpha+r}, \beta_{2,\alpha+r}, \ldots, \beta_{m+1,\alpha+r} \) are the \((m+1)\) positive roots of the equation \( G(x) = \alpha + r \), and \( d := (d_1, d_2, \ldots, d_{m+1})' \) is the unique solution of the linear system \( Ad = 1 \), where \( A \) associated with \( \alpha + r \) is defined in Theorem 3.3 and \( 1 = (1, 1, \ldots, 1)' \).

Proof. See Appendix A. \( \square \)

To invert the Laplace transform, we employ one-sided, one-dimensional Euler inversion method (see [1]). The corresponding numerical results are given in the upper panels of Table 2 and 3, where EI value and MC value represent the results obtained via the Euler inversion method and the Monte Carlo simulation, respectively. Std Err is the associated standard error.
### Pricing lookback options under the MEM

<table>
<thead>
<tr>
<th>$M$</th>
<th>$\lambda$</th>
<th>EI value</th>
<th>MC value</th>
<th>Std Err</th>
<th>EI value</th>
<th>MC value</th>
<th>Std Err</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>7.13774</td>
<td>7.14438</td>
<td>0.00717</td>
<td>15.11777</td>
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</tr>
<tr>
<td>105</td>
<td>3</td>
<td>8.38446</td>
<td>8.39709</td>
<td>0.00737</td>
<td>15.96876</td>
<td>15.97562</td>
<td>0.00679</td>
</tr>
<tr>
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<td>9.54401</td>
<td>9.55256</td>
<td>0.00747</td>
<td>16.79455</td>
<td>16.80643</td>
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</tr>
<tr>
<td>107</td>
<td>3</td>
<td>8.91452</td>
<td>8.93285</td>
<td>0.01084</td>
<td>16.33422</td>
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</tr>
<tr>
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<td>10.06516</td>
<td>0.01108</td>
<td>17.15486</td>
<td>17.16885</td>
<td>0.01052</td>
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</tr>
</tbody>
</table>

### Pricing up-and-in call barrier options under the MEM when $K$ varies and $H = 115$

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>$K$</th>
<th>EI value</th>
<th>MC value</th>
<th>Std Err</th>
<th>EI value</th>
<th>MC value</th>
<th>Std Err</th>
</tr>
</thead>
<tbody>
<tr>
<td>101</td>
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<td>10.03530</td>
<td>0.02077</td>
<td>13.94197</td>
<td>13.91411</td>
<td>0.02840</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>105</td>
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<td>0.02072</td>
<td>12.25099</td>
<td>12.24949</td>
<td>0.02935</td>
</tr>
<tr>
<td>109</td>
<td>108.4158</td>
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<td>0.02073</td>
<td>10.70121</td>
<td>10.67303</td>
<td>0.02984</td>
<td></td>
</tr>
</tbody>
</table>

### Pricing up-and-in call barrier options under the MEM when $H$ varies and $K = 102$

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>$H$</th>
<th>EI value</th>
<th>MC value</th>
<th>Std Err</th>
<th>EI value</th>
<th>MC value</th>
<th>Std Err</th>
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</thead>
<tbody>
<tr>
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<tr>
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<td>0.02877</td>
</tr>
<tr>
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</tbody>
</table>

Table 2: Pricing lookback and barrier options under the MEM. The Euler inversion (EI value) vs. Monte Carlo simulation (MC value). Default parameters are $r = 0.05$, $S_0 = 100$, $\eta_1 = 30$, $\eta_2 = 50$, $\theta_1 = 30$, $\theta_2 = 40$, $p_u = 0.4$, $q_d = 0.6$, $p_1 = 1.2$, $p_2 = -0.2$, $q_1 = 1.3$, $q_2 = -0.3$ and $t = 1$. EI values are obtained using Euler inversion (related parameters are $A = 18$ and $n = 30$ for lookback options and $A_1 = A_2 = 18$, $n_1 = 30$, $n_2 = 50$ and $X = 1000$ for barrier options). MC values are Monte Carlo simulation estimates by simulating 20,000 sample paths and using step size 0.00001 for lookback options and by simulating 100,000 sample paths and using step size 0.00005 for barrier options. We can see that all the EI values stay within the 95% confidence intervals of the associated MC values. The CPU time for generating one EI value for lookback options, one MC value for lookback options, one EI value for barrier options and one MC value for barrier options are around 0.04 seconds, 3 minutes, 6 seconds and 2 minutes, respectively.

Furthermore, the numerical inversion is robust w.r.t. the inversion algorithm parameters. For example, any $A \in [15, 45]$ produces almost identical results with four-digit accuracy.

\[^4\] Furthermore, the numerical inversion is robust w.r.t. the inversion algorithm parameters. For example, any $A \in [15, 45]$ produces almost identical results with four-digit accuracy.
### Deltas of lookback options under the MEM

<table>
<thead>
<tr>
<th>$S_0$</th>
<th>$\sigma$</th>
<th>EI value</th>
<th>MC value</th>
<th>Std Err</th>
<th>EI value</th>
<th>MC value</th>
<th>Std Err</th>
</tr>
</thead>
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<td>0.15430</td>
<td>-0.15790</td>
<td>0.00294</td>
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</tr>
<tr>
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<td>-0.00711</td>
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<td>0.00201</td>
<td>0.00181</td>
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</tr>
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</tr>
<tr>
<td>0.2</td>
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<tr>
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<td>0.05264</td>
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</tr>
<tr>
<td>0.4</td>
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### Deltas of up-and-in call barrier options under the MEM

<table>
<thead>
<tr>
<th>$S_0$</th>
<th>$K$</th>
<th>$\sigma = 0.2$</th>
<th>EI value</th>
<th>MC value</th>
<th>Std Err</th>
<th>$\sigma = 0.3$</th>
<th>EI value</th>
<th>MC value</th>
<th>Std Err</th>
</tr>
</thead>
<tbody>
<tr>
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<td>0.66375</td>
<td>0.00230</td>
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</tr>
<tr>
<td>100</td>
<td>0.60236</td>
<td>0.60289</td>
<td>0.00242</td>
<td>0.61359</td>
<td>0.61711</td>
<td>0.00237</td>
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<td></td>
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<td>0.56205</td>
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<td></td>
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<td></td>
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<tr>
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<td>0.58646</td>
<td>0.58203</td>
<td>0.00241</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 3: Deltas of lookback and barrier options under the MEM. The Euler inversion (EI value) vs. Monte Carlo simulation (MC value). The default choices of unvarying parameters are $r = 0.05$, $M = 110$ (for lookback options), $H = 110$ and $\lambda = 3$ (for barrier options), $\eta_1 = 30$, $\eta_2 = 50$, $\theta_1 = 30$, $\theta_2 = 40$, $p_u = 0.4$, $q_d = 0.6$, $p_1 = 1.2$, $p_2 = -0.2$, $q_1 = 1.3$, $q_2 = -0.3$ and $t = 1$. EI values are obtained using the Euler inversion with parameters $A = 18$ and $n = 30$ for lookback options and $A_1 = A_2 = 18$, $n_1 = 30$, $n_2 = 50$ and $X = 1000$ for barrier options. All the MC values along with the associated standard errors (denoted by Std Err) are obtained by simulating 15,000 sample paths and by using step size 0.00001. We can see that all the EI values stay within the 95% confidence intervals of the associated MC values. The CPU times to generate one EI value for lookback option delta, one MC value for lookback option delta, one EI value for barrier option delta and one MC value for barrier option delta are around 0.04 seconds, 2 minutes, 6 seconds and 2 minutes, respectively.

### 5.2 Barrier Options

There are eight types of (one dimensional, single) barrier options: up (down)-and-in (out) call (put) options. Here, we only illustrate how to deal with the up-and-in call barrier option (UIC) because the other seven barrier options can be priced similarly. The price of a UIC with a fixed strike $K$ and a maturity $T$ under the risk neutral measure $P$ can be expressed as

$$E[e^{-rT}(S_T - K)^+I\{\tau_b < T\}],$$

where $H > S_0$ is the barrier level and $b = \log(H/S_0)$ is the barrier corresponding to the return process $X_t \equiv \log(S_t/S_0)$. To apply the two-sided Euler inversion method proposed by Petrella [41], we introduce a scaling factor $X$ so that the price of a UIC can be rewritten as

$$UIC(k, T) = E\left[X e^{-rT} \left(\frac{S_T}{X} - e^{-k}\right)^+ I\{\tau_b < T\}\right].$$

(21)
where \( k = \log(X/K) \). Note that the scaling factor is crucial in our algorithm as it ensures that the Euler inversion method converges rapidly, making the algorithm accurate and efficient.

Define \( \hat{f}_{UIC}(\alpha, \psi) \) and \( \hat{\Delta}_{UIC}(\alpha, \psi) \) as the double Laplace transforms of the price \( UIC(k, T) \) in (21) and the delta \( \Delta_{UIC}(k, T) := \frac{\partial UIC(k, T)}{\partial S_0} \) w.r.t. \( T \) and \( k \), respectively, i.e.,

\[
\hat{f}_{UIC}(\alpha, \psi) := \int_{0}^{+\infty} \int_{-\infty}^{+\infty} e^{-\psi k - \alpha T} UIC(k, T) dk dT,
\]

\[
\hat{\Delta}_{UIC}(\alpha, \psi) := \int_{0}^{+\infty} \int_{-\infty}^{+\infty} e^{-\psi k - \alpha T} \Delta_{UIC}(k, T) dk dT.
\]

**Theorem 5.2** For any \( \psi \in (0, \eta_1 - 1) \) and sufficiently large \( \alpha > \max(G(\psi + 1) - r, 0) \),

\[
\hat{f}_{UIC}(\alpha, \psi) = S_0^{\psi + 1} \sum_{l=1}^{m+1} w_l e^{-\beta_{l,\alpha+r} b} X^{\psi}(\psi + 1)(r + \alpha - G(\psi + 1)) \tag{22}
\]

and

\[
\hat{\Delta}_{UIC}(\alpha, \psi) = S_0^{\psi} \sum_{l=1}^{m+1} w_l e^{-\beta_{l,\alpha+r} b} X^{\psi}(r + \alpha - G(\psi + 1)), \tag{23}
\]

where \( \beta_{1,\alpha+r}, \beta_{2,\alpha+r}, \ldots, \beta_{m+1,\alpha+r} \) are the \( (m+1) \) positive roots of the equation \( G(x) = \alpha + r \), and \( w := (w_1, w_2, \ldots, w_{m+1})' \) is the unique solution of the linear system \( Aw = J \), where \( A \) associated with \( \alpha + r \) is defined as in Theorem 3.3 and \( J = e^{(\psi + 1) b}(1, \frac{m_1}{\eta_1 - \psi - 1}, \frac{m_2}{\eta_2 - \psi - 1}, \ldots, \frac{m_m}{\eta_m - \psi - 1})' \).

**Proof.** See Appendix A. □

To compute the prices and deltas of barrier options, a two-sided, two-dimensional Euler inversion method is applied (see Petrella [41]). The numerical results are given in the lower panels of Table 2 and Table 3, where “EI value”, “MC value” and “Std Err” have the same meanings as for lookback options. For Monte Carlo simulation, we use \( S_T \) as a control variate to achieve variance reduction. All the EI values stay within the 95% confidence intervals of the associated MC values. The CPU time of computing one price via the Euler inversion is around 6.0 seconds, while the CPU time of generating one MC value is around 2 minutes.

### 6 An Example of Approximating Merton’s Model via the MEM

Although Merton’s normal jump diffusion model (see [40]) is very popular in finance, analytical pricing of path-dependent options under Merton’s model remains challenging. We shall approximate Merton’s model by the MEM, partly due to the denseness of the mixed-exponential distributions w.r.t. the class of all the distributions in the sense of weak convergence. In this case, the numerical inversion is also insensitive to the change of the algorithm parameters \( A_1, A_2 \) and \( X \). Indeed, any \( A_1 \in [15, 55], A_2 \in [16, 45] \) and \( X \in [300, 4000] \) can produce almost identical results with four-digit accuracy.
section, our objective is not to discuss how to approximate Merton’s model optimally, which by itself is an interesting open problem. Rather, we shall provide a simple example to illustrate this approximation may lead to quite accurate European, lookback, and barrier option prices and deltas for Merton’s model.

For simplicity, we intend to approximate Merton’s model with the jump size distribution \( N(0, 0.01^2) \), the normal distribution with mean 0 and standard deviation 0.01, using the MEM with the pdf of the jump size given by

\[
f_Y(x) = 0.5 \left( 8.7303 \times 213.0215 e^{-213.0215|x|} + 2.1666 \times 236.0406 e^{-236.0406|x|} \right. \\
\left. -10 \times 237.1139 e^{-237.1139|x|} + 0.0622 \times 939.7441 e^{-939.7441|x|} + 0.0409 \times 939.8021 e^{-939.8021|x|} \right) .
\]  

(24)

These parameters are obtained by minimizing the sum of the square differences between cdf values of \( N(0, 0.01^2) \) and the mixed-exponential distribution over the grid points on the interval \([-0.035, 0.035]\). We first select 100 best starting points among around 1,600,000 points and then minimize the objective function by starting from each of these 100 points. The final solution is the best one among the 100 optimal solutions. Figure 4 demonstrates the close fit of the mixed-exponential distribution (24) to the cdf of \( N(0, 0.01^2) \).

![Approximate the cdf of Normal(0,0.01²)](image)

Figure 4: Approximate the normal distribution \( N(0,0.01^2) \) using the mixed-exponential distribution with the pdf given by (24).

Using the MEM with jump size pdf being (24), we shall price European, lookback and barrier options as well as calculate associated deltas approximately under the Merton’s model.
Table 4 provides the approximation to the European option prices and deltas. Our approximation appears to be reasonably good, because the maximum absolute errors between our approximate values (denoted by EI values) and “True values” are quite small.

<table>
<thead>
<tr>
<th>Prices of European options under Merton’s model</th>
<th>λ = 1</th>
<th>λ = 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>σ</td>
<td>K</td>
<td>EI value</td>
</tr>
<tr>
<td>95</td>
<td>13.35534</td>
<td>13.35476</td>
</tr>
<tr>
<td>0.2</td>
<td>100</td>
<td>10.46061</td>
</tr>
<tr>
<td>105</td>
<td>8.03196</td>
<td>8.03126</td>
</tr>
<tr>
<td>95</td>
<td>16.80753</td>
<td>16.80711</td>
</tr>
<tr>
<td>0.3</td>
<td>100</td>
<td>14.23802</td>
</tr>
<tr>
<td>105</td>
<td>11.98391</td>
<td>11.98345</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Deltas of European options under Merton’s model</th>
<th>λ = 1</th>
<th>λ = 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>σ</td>
<td>K</td>
<td>EI value</td>
</tr>
<tr>
<td>95</td>
<td>0.72772</td>
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<tr>
<td>0.2</td>
<td>100</td>
<td>0.63676</td>
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<tr>
<td>105</td>
<td>0.54228</td>
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<tr>
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<td>0.68706</td>
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<tr>
<td>0.3</td>
<td>100</td>
<td>0.62425</td>
</tr>
<tr>
<td>105</td>
<td>0.56124</td>
<td>0.56124</td>
</tr>
</tbody>
</table>

Table 4: Pricing European options and calculating deltas under Merton’s model by approximating it with the MEM. Default parameters are: \( r = 0.05 \), \( S_0 = 100 \) and \( t = 1 \). “EI” values and “True” values are obtained by using Euler inversion (related parameters are \( A = 18 \), \( n = 30 \), and \( X = 10000 \)) under the approximate MEM with the jump size pdf being (24) and under Merton’s model, respectively.

Table 5 and Table 6 provides approximate lookback and barrier option prices and deltas under Merton’s model, respectively. All of our numerical approximations (denoted by EI values) obtained using the approximate MEM stay within the 95% confidence intervals of the Monte Carlo simulation estimates (denoted by MC values) obtained under Merton’s model. For Monte Carlo simulation, we use \( S_T \) as a control variate to achieve variance reduction. In addition, our approximation method is very fast in that it takes only around 0.04 seconds and 6 seconds to produce one EI value for lookback options and barrier options, respectively.

7 Conclusion

We propose a jump diffusion model for option pricing whose jump sizes have the mixed-exponential distribution, which can approximation any jump size distribution. The Laplace transforms of option prices and deltas for some path-dependent options such as lookback and
Pricing lookback options under Merton’s model

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>$M$</th>
<th>$\sigma$</th>
<th>EI value</th>
<th>MC value</th>
<th>Std Err</th>
<th>EI value</th>
<th>MC value</th>
<th>Std Err</th>
</tr>
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<tr>
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<td>15.64581</td>
<td>0.02275</td>
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</tr>
</tbody>
</table>

Table 5: Pricing lookback and barrier options under Merton’s model by approximating it with the MEM. The Euler inversion (EI value) vs. Monte Carlo simulation (MC value). Default parameters are $r = 0.05$, $S_0 = 100$ and $t = 1$. EI values are obtained using the Euler inversion (related parameters are $A = 18$ and $n = 30$ for lookback options and $A_1 = A_2 = 18$, $n_1 = 30$, $n_2 = 50$ and $X = 1000$ for barrier options) under the approximate MEM with the jump size pdf being (24). MC values are Monte Carlo simulation estimates under Merton’s model by simulating 100,000 sample paths and by using step size 0.00001 for lookback options and 0.00005 for barrier options. We can see that all the EI values obtained using the approximate MEM stay within the 95% confidence intervals of the MC values obtained under Merton’s model. The CPU times to generate one EI value for lookback option price, one MC value for lookback option price, one EI value for barrier option price and one MC value for barrier option price are around 0.04 seconds, 10 minutes, 6 seconds and 2 minutes, respectively.

Pricing up-and-in call barrier options under Merton’s model when $K$ varies and $H = 115$

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>$K$</th>
<th>$\sigma$</th>
<th>EI value</th>
<th>MC value</th>
<th>Std Err</th>
<th>EI value</th>
<th>MC value</th>
<th>Std Err</th>
</tr>
</thead>
<tbody>
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<tr>
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<td>0.02754</td>
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<td>0.01839</td>
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<td>13.29997</td>
<td>0.02754</td>
</tr>
<tr>
<td>1</td>
<td>115</td>
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<td>9.14142</td>
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<td>13.20890</td>
<td>13.25942</td>
<td>0.02804</td>
</tr>
</tbody>
</table>

Pricing up-and-in call barrier options under Merton’s model when $H$ varies and $K = 102$

<table>
<thead>
<tr>
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<th>$H$</th>
<th>$\sigma$</th>
<th>EI value</th>
<th>MC value</th>
<th>Std Err</th>
<th>EI value</th>
<th>MC value</th>
<th>Std Err</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>105</td>
<td>0.2</td>
<td>9.45243</td>
<td>9.44670</td>
<td>0.01839</td>
<td>13.31261</td>
<td>13.29997</td>
<td>0.02754</td>
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<tr>
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<tr>
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<td>9.14142</td>
<td>9.12743</td>
<td>0.01979</td>
<td>13.20890</td>
<td>13.25942</td>
<td>0.02804</td>
</tr>
</tbody>
</table>

In addition, we show that the mixed-exponential jump diffusion model may be used to approximate Merton’s normal jump diffusion model. Open problems for future research include pricing of sequential barrier options and finite-horizon American options under the mixed-exponential jump diffusion model, as well as extensions to more general models, e.g., the models with stochastic interest rates.
Deltas of lookback options under Merton’s model

<table>
<thead>
<tr>
<th>$S_0$</th>
<th>$\sigma$</th>
<th>EI value</th>
<th>MC value</th>
<th>Std Err</th>
<th>EI value</th>
<th>MC value</th>
<th>Std Err</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
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<td>-0.17610</td>
<td>0.00406</td>
<td>-0.17579</td>
<td>-0.18082</td>
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<tr>
<td>100</td>
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<td>0.13043</td>
<td>0.00431</td>
<td>0.12709</td>
<td>0.12681</td>
<td>0.00431</td>
</tr>
<tr>
<td>102</td>
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</tr>
</tbody>
</table>

Deltas of up-and-in call barrier options under Merton’s model

<table>
<thead>
<tr>
<th>$S_0$</th>
<th>$K$</th>
<th>EI value</th>
<th>MC value</th>
<th>Std Err</th>
<th>EI value</th>
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<tr>
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</tr>
<tr>
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<td>0.55839</td>
<td>0.55790</td>
<td>0.00095</td>
<td></td>
</tr>
</tbody>
</table>

Table 6: Deltas of lookback and barrier options under Merton’s model by approximating it with the MEM with the jump size pdf being (24). The Euler inversion (EI value) vs. Monte Carlo simulation (MC value). Default parameters are $M = 110$ (for lookback options), $H = 110$ and $\lambda = 3$ (for barrier options), $r = 0.05$, and $t = 1$. Parameters for the Euler inversion methods are $A_1 = A_2 = 18$, $n_1 = 30$, $n_2 = 50$ and $X = 1000$ for the barrier options. MC values along with the associated standard errors (denoted by Std Err) are obtained by simulating 100,000 sample paths and by using 20,000 steps for barrier options and by simulating 10,000 sample paths and by using 150,000 steps for lookback options. We can see that all the EI values obtained using the approximate MEM stay within the 95% confidence intervals of the MC values obtained under Merton’s model. The CPU times to generate one EI value for lookback option delta, one MC value for lookback option delta, one EI value for barrier option delta and one MC value for barrier option delta are around 0.04 seconds, 2 minutes, 6 seconds and 2 minutes, respectively.

Acknowledgments

We thank Gerard Cachon, Michael Fu, the Associate Editor, two anonymous referees, and participants of INFORMS annual conferences and various seminars at Columbia University, Cornell University, University of Oxford, Johns Hopkins University, Georgia Tech, and Hong Kong University of Science and Technology for their helpful comments. The research of the first author is partially supported by the General Research Fund (GRF) of Hong Kong RGC (Project Reference No. 610709) and DAG11EG07G. The research of the second author is supported in part by the National Science Foundation.
A Some Proofs

Proof of Theorem 5.1. First of all, define $M_X(t) := \max_{0 \leq u \leq t} X_u$. Then for any $t > 0$, we shall prove that

$$\lim_{y \to +\infty} e^y P[M_X(t) \geq y] = 0. \quad (25)$$

Indeed, note that the process $\{e^{\theta X_t - G(\theta)t} : t \geq 0\}$ is a martingale for any $\theta \in (-\theta_1, \eta_1)$ since $G(\theta)$ is the exponent of the Lévy process $\{X_t : t \geq 0\}$. Fix $\theta \in (1, \eta_1)$ such that $G(\theta) > 0$. This $\theta$ must exist because $G(\eta_1) = +\infty$ and $G(\theta)$ is continuous on the interval $(1, \eta_1)$. Note that

$$e^{\theta y} P(\tau_y \leq t) \leq E[e^{\theta X_{\tau_y}}] \leq e^{G(\theta)t} E[e^{\theta X_{\tau_y} - G(\theta)(t \wedge \tau_y)}] \leq e^{G(\theta)t},$$

where the last inequality holds thanks to the optional sampling theorem. So for any $y > 0$,

$$e^{\theta y} P[M_X(t) \geq y] = e^{(1-\theta)y} e^{\theta y} P[M_X(t) \geq y] = e^{(1-\theta)y} e^{\theta y} P(\tau_y \leq t) \leq e^{(1-\theta)y} e^{G(\theta)t}.$$

Note that $\theta > 1$, so letting $y$ go to infinity completes the proof of (25).

Next, define

$$L(S_0, M, T) := E\left[e^{-rT} \max \left\{ M, \max_{0 \leq t \leq T} S_t \right\} \right] = E\left[e^{-rT} \max \left\{ M, S_0 e^{M_X(T)} \right\} \right],$$

and $z := \log(M/S_0) \geq 0$. Then we have

$$L(S_0, M, T) = S_0 E\left[e^{-rT} \max \left\{ e^z, e^{M_X(T)} \right\} \right]$$

$$= S_0 e^{-rT} E\left[\left(e^{M_X(T)} - e^z\right) I_{\{M_X(T) \geq z\}}\right] + S_0 e^{z-rT}$$

$$= S_0 e^{-rT} E\left[\left(e^{M_X(T)} - e^z\right) I_{\{M_X(T) \geq z\}}\right] + M e^{-rT}.$$

On the other hand, we can obtain

$$E\left[\left(e^{M_X(T)} - e^z\right) I_{\{M_X(T) \geq z\}}\right] = \int_0^{+\infty} (e^y - e^z) I_{\{y \geq z\}} f_{M_X(T)}(y) dy$$

$$= - \int_z^{+\infty} (e^y - e^z) dP(M_X(T) \geq y)$$

$$= \int_z^{+\infty} e^y P(M_X(T) \geq y) dy,$$

where $f_{M_X(T)}$ is the pdf of $M_X(T)$ and the third equality holds because of (25). Therefore,

$$L(S_0, M, T) = S_0 e^{-rT} \int_z^{+\infty} e^y P(M_X(T) \geq y) dy + M e^{-rT}.$$
For any $\alpha > 0$, the Laplace transform of $L(S_0, M, T)$ w.r.t. $T$ is given by
\[
\int_0^+ e^{-\alpha T} L(S_0, M, T) dT = S_0 \int_0^+ e^{-\alpha T} e^{-rT} \int_0^{+\infty} e^y P(M_X(T) \geq y) dy dT + \frac{M}{\alpha + r}.
\]
\[
= S_0 \int_0^+ e^y \left[ \int_0^{+\infty} e^{-(\alpha+r)T} P(M_X(T) \geq y) dT \right] dy + \frac{M}{\alpha + r}. \tag{26}
\]
Note that for any $y > z \geq 0$, integration by parts leads to
\[
\int_0^{+\infty} e^{-(\alpha+r)T} P(M_X(T) \geq y) dT = \frac{1}{\alpha + r} \int_0^{+\infty} e^{-(\alpha+r)T} dP(M_X(T) \geq y) = \frac{1}{\alpha + r} E \left[ e^{-(\alpha+r)\tau_y} \right].
\]
Applying (13) with $x = 0$, we have that for sufficiently large $\alpha > 0$,
\[
\int_0^{+\infty} e^{-(\alpha+r)T} P(M_X(T) \geq y) dT = \frac{1}{\alpha + r} \sum_{i=1}^{m+1} d_i e^{-\beta_{i,\alpha+r} y} \tag{27}
\]
Plugging (27) into (26) yields
\[
\int_0^{+\infty} e^{-\alpha T} L(S_0, M, T) dT = S_0 \int_0^{+\infty} e^y \left[ \frac{1}{\alpha + r} \sum_{i=1}^{m+1} d_i e^{-\beta_{i,\alpha+r} y} \right] dy + \frac{M}{\alpha + r}
\]
\[
= \frac{S_0}{\alpha + r} \sum_{i=1}^{m+1} d_i \int_0^{+\infty} e^{-(\beta_{i,\alpha+r} - 1) y} dy + \frac{M}{\alpha + r}.
\]
Note that $\beta_{1,\alpha+r} > \beta_{1,r} = 1$ and $\beta_{i,\alpha+r} > \eta_i > 1$ for any $i = 2, \cdots, m + 1$. So we have that
\[
\int_0^{+\infty} e^{-\alpha T} L(S_0, M, T) dT = \frac{S_0}{\alpha + r} \sum_{i=1}^{m+1} \frac{d_i}{\beta_{i,\alpha+r} - 1} e^{-(\beta_{i,\alpha+r} - 1) z} + \frac{M}{\alpha + r}
\]
\[
= \frac{S_0}{\alpha + r} \sum_{i=1}^{m+1} \frac{d_i}{\beta_{i,\alpha+r} - 1} \left( \frac{S_0}{M} \right)^{\beta_{i,\alpha+r} - 1} + \frac{M}{\alpha + r},
\]
which leads to (19) since $L_P(T) = L(S_0, M, T) - S_0$. Then (20) can be obtained by interchanging derivatives and integrals based on Theorem A. 12 on pp. 203-204 in Schiff [43]. □
Proof of Theorem 5.2. Note that \( \hat{f}_{UIC}(\alpha, \psi) \) can be expressed as follows.

\[
\hat{f}_{UIC}(\alpha, \psi) = X \int_{0}^{+\infty} \int_{-\infty}^{+\infty} e^{-\psi k - (r + \alpha)T} e^{\left(\frac{S_T}{X} - e^{-k}\right)^+} I_{(\tau_b < T)} \, dk dT
\]

\[
= X E \left[ \int_{0}^{+\infty} e^{-(r + \alpha)T} I_{(\tau_b < T)} \int_{-\log(S_T/X)}^{+\infty} e^{-\psi k} \left(\frac{S_T}{X} - e^{-k}\right)^+ \, dk dT \right]
\]

\[
= \frac{1}{\psi(\psi + 1)X^\psi} E \left[ \int_{0}^{+\infty} e^{-(r + \alpha)T} I_{(\tau_b < T)} S_T^{\psi + 1} \, dT \right]
\]

\[
= \frac{1}{\psi(\psi + 1)X^\psi} \int_{\tau_b}^{+\infty} e^{-(r + \alpha)T} S_T^{\psi + 1} \, dT
\]

where the last equality holds due to a change of variables \( T = t + \tau_b \). On the other hand, the strong Markov property of the return process \( \{X_t\} \) implies that for any \( \alpha > G(\psi + 1) - r \),

\[
E \left\{ e^{-(r + \alpha)\tau_b} \int_{0}^{+\infty} e^{-(r + \alpha)t} S_t^{\psi + 1} \, dt \mid \mathcal{F}_{\tau_b} \right\} = e^{-(r + \alpha)\tau_b} S_{\tau_b}^{\psi + 1} \int_{0}^{+\infty} e^{-(r + \alpha)t} E[e^{(\psi + 1)X_t}] \, dt
\]

\[
= e^{-(r + \alpha)\tau_b} S_{\tau_b}^{\psi + 1} \int_{0}^{+\infty} e^{-(r + \alpha - G(\psi + 1))t} \, dt
\]

\[
= \frac{S_0^{\psi + 1} e^{-(r + \alpha)\tau_b + (\psi + 1)X_{\tau_b}}}{r + \alpha - G(\psi + 1)},
\]

where \( G(\cdot) \) is the exponent of \( \{X_t\} \). Combining them together and applying (9) with \( x = 0 \) yields (22) immediately. Then (23) can be obtained by interchanging derivatives and integrals based on Theorem A. 12 on pp. 203-204 in Schiff [43]. □

B Hyper-Exponential Distributions and Completely Monotone Distributions

A distribution with the pdf \( h(x) \) for \( x \geq 0 \) is completely monotone if the function \( h(x) \) is completely monotone, i.e., \( h^{(k)}(x) \) exists for any \( k \geq 1 \) and \( (-1)^k h^{(k)}(x) \geq 0 \) for any \( x > 0 \) and \( k \geq 1 \) (see, e.g., [25]). A distribution with the pdf \( h(x) \) for \( x \in (-\infty, +\infty) \) is completely monotone if the two functions \( h(x)I_{[x \geq 0]} \) and \( h(-x)I_{[x \geq 0]} \) are both completely monotone. Without loss of generality, from now on we assume that the supports of all the cdf’s are \((0, +\infty)\).

For any completely monotone distribution with the cdf \( F(x) \), there must exist a sequence of hyper-exponential distributions that converge to \( F(x) \) weakly (see, e.g., p. 256 in [25]). The following proposition shows the converse under some conditions.
Proposition B.1 Consider a sequence of hyper-exponential distributions (with the cdf’s \{F_n(x)\} and the pdf’s \{f_n(x)\}), which converge to a continuous distribution (with the cdf \(F(x)\) and the pdf \(f(x)\) weakly, namely \(\lim_{n \to +\infty} F_n(x) = F(x)\) for any \(x > 0\). Assume that (i) \(f^{(k)}(x)\) exists for any \(x > 0\) and \(k \geq 1\); and (ii) \(\lim_{n \to +\infty} f^{(k)}_n(x) = f^{(k)}(x)\) for any \(x > 0\) and \(k \geq 1\). Then \(f(x)\) is completely monotone.

Proof. Since the hyper-exponential distribution is completely monotone, it follows that \((-1)^k f^{(k)}_n(x) \geq 0\) for any \(x > 0\) and \(k, n \geq 1\). Then by assumption (ii), we have \((-1)^k f^{(k)}(x) = \lim_{n \to +\infty} (-1)^k f^{(k)}_n(x) \geq 0\) for any \(x > 0\) and \(k \geq 1\), which implies that \(f(x)\) is completely monotone. □

References


A Proof of Theorem 3.1

Proof. Without loss of generality, we assume that \( p_u > 0, q_d > 0, p_i \neq 0 \) for \( i = 1, \cdots, m \), and \( q_j \neq 0 \) for \( j = 1, \cdots, n \). First of all, it is easily seen that \( G(x) - \alpha \) has the same roots as 
\[
\left(G(x) - \alpha\right) \prod_{i=1}^{m} (x - \eta_i) \prod_{j=1}^{n} (x + \theta_j),
\]
which is a polynomial with order \( m + n + 2 \). This implies that for any \( \alpha \in \mathbb{R} \), the function \( G(x) = \alpha \) has at most \( (m + n + 2) \) real roots. From now on we shall show that for sufficiently large \( \alpha > 0 \), the function has exactly \( (m + n + 2) \) real roots, among which \( m + 1 \) are positive and \( n + 1 \) are negative. Due to the symmetry, we will focus only on arguing that for sufficiently large \( \alpha > 0 \), the function has \( (m + 1) \) positive roots.

Note that there exist \( m \) positive singularities \( \eta_1, \cdots, \eta_m \) for the function \( G(x) \), which divide the positive real line into \( (m + 1) \) disjoint intervals: \((0, \eta_1), (\eta_1, \eta_2), \cdots, (\eta_{m-1}, \eta_m), (\eta_m, +\infty)\). Since \( G(\eta_1^-) = +\infty \) (because \( p_1 > 0 \)) and \( G(0) - \alpha = -\alpha \), we know that for any \( \alpha > 0 \), \( G(x) = \alpha \) has at least one real root on the interval \((0, \eta_1)\). We plan to show there exist \( m \) real roots on the other \( m \) intervals for sufficiently large \( \alpha > 0 \).

For convenience, we define \( p_{m+1} := +\infty, \eta_{m+1} = +\infty \), and two sets \( S^+ \) and \( S^- \) as follows
\[
S^+ := \{i \in \{1, \cdots, m\} : p_i > 0 \text{ and } p_{i+1} < 0\},
\]
\[
S^- := \{i \in \{1, \cdots, m\} : p_i < 0 \text{ and } p_{i+1} > 0\}.
\]
Noting that \( p_1 > 0 \) and \( p_{m+1} = +\infty > 0 \), we can easily see that the number of elements in \( S^+ \) and that in \( S^- \) are identical. Moreover, if the number of elements is \( k > 0 \) and if we assume
\[
S^+ = \{i_1^+, \cdots, i_k^+\} \quad \text{and} \quad S^- = \{i_1^-, \cdots, i_k^-\},
\]
we must have \( i_1^- < i_2^- < i_1^+ < i_2^+ < \cdots < i_k^- < i_k^+ \). In other words, the elements in \( S^+ \) and those in \( S^- \) are arranged alternatingly.

We categorize the \( m \) intervals, \((\eta_1, \eta_2), \cdots, (\eta_{m-1}, \eta_m), (\eta_m, \eta_{m+1})\), into three types.

Type I: \((\eta_i, \eta_{i+1})\) with \( i \in S^+ \). Type II: \((\eta_i, \eta_{i+1})\) with \( i \in S^- \). Type III: \((\eta_i, \eta_{i+1})\) with \( i \notin S^+ \cup S^- \).

Then we will show that

(1) If \( S^+ \neq \emptyset \), then for sufficiently large \( \alpha > 0 \), \( G(x) = \alpha \) has no real roots on any interval of Type I:
(2) If $S^- \neq \emptyset$, then for sufficiently large $\alpha > 0$, $G(x) = \alpha$ has at least two real roots on any interval of Type II;

(3) If $S^+ \cup S^- \neq \{1, \cdots, m\}$, then for any $\alpha > 0$, $G(x) = \alpha$ has at least one real root on any interval of Type III.

In fact, (1) is implied by the fact that $G(\eta_i +) = G(\eta_{i+1} -) = -\infty$ due to $p_i > 0$ and $p_{i+1} < 0$. Similarly, (2) is correct because $G(\eta_i +) = G(\eta_{i+1} -) = +\infty$. As for (3), if $i \notin S^+ \cup S^-$, we have either $p_i > 0$ and $p_{i+1} > 0$, or $p_i < 0$ and $p_{i+1} < 0$. In the former case, we obtain $G(\eta_i +) = -\infty$ and $G(\eta_{i+1} -) = +\infty$, while in the latter case, we obtain $G(\eta_i +) = +\infty$ and $G(\eta_{i+1} -) = -\infty$. Thus (3) follows immediately. Recall that the number of elements in $S^+$ and that in $S^-$ are identical. Accordingly, the number of intervals of Type I and that of Type II are identical, too. So for sufficiently large $\alpha > 0$, the equation $G(x) = \alpha$ must have $m$ roots on the $m$ intervals, $(\eta_1, \eta_2), \cdots, (\eta_{m-1}, \eta_m), (\eta_m, \eta_{m+1})$. The proof is finished. □

**B Proof of Theorem 3.2**

**Proof.** First of all, it suffices to show that Theorem 3.2 holds when $u(x)$ solves the OIDE $(Lu)(x) = \alpha u(x)$ for any $x \in \mathbb{R}$, because all the terms involving the function $g$ will disappear eventually. We give an example to describe this point more explicitly.

Consider a special case $m = 1$ and $n = 1$, i.e., a double exponential jump diffusion process. Then the OIDE (6) is reduced to

$$\frac{\sigma^2}{2} u''(x) + \mu u'(x) - (\lambda + \alpha) u(x)$$

$$+ \lambda \left[q \int_{-\infty}^{0} u(x+y)e^{\theta_1 y} dy + p_a \eta_1 \int_{0}^{x_0-x} u(x+y)e^{-\eta_1 y} dy + p_a \eta_1 \int_{x_0-x}^{+\infty} g(x+y)e^{-\eta_1 y} dy \right] = 0.$$

A transformation $z = x + y$ leads to

$$\frac{\sigma^2}{2} u''(x) + \mu u'(x) - (\lambda + \alpha) u(x)$$

$$+ \lambda q \int_{-\infty}^{x} u(z)e^{\theta_1 z} dz + \lambda p_a \eta_1 e^{-\eta_1 z} \left[ \int_{x}^{x_0} u(z)e^{-\eta_1 z} dz + C_g \right] = 0,$$

where $C_g := \int_{x_0}^{+\infty} g(z)e^{-\eta_1 z} dz$ is a constant. Then multiplying both sides by $e^{-\eta_1 z}$ gives

$$\frac{\sigma^2}{2} e^{-\eta_1 z} u''(x) + \mu e^{-\eta_1 z} u'(x) - (\lambda + \alpha) e^{-\eta_1 z} u(x)$$

$$+ \lambda q \int_{-\infty}^{x} u(z)e^{\theta_1 z} dz + \lambda p_a \eta_1 \left[ \int_{x}^{x_0} u(z)e^{-\eta_1 z} dz + C_g \right] = 0.$$
Finally taking a derivative w.r.t. $x$ and then multiplying both sides by $e^{\eta x}$, we have
\[
\frac{\sigma^2}{2}u''(x) + \left(-\frac{\sigma^2}{2}\eta_1 + \mu\right)u''(x) + (-\mu p_1 \eta_1 - \lambda - \alpha)u'(x) + (\alpha \eta_1 + \lambda q_1 + \lambda q_0 \eta_1)u(x)
\]
\[-\lambda q_0 \eta_1 (\theta_1 + \eta_1) e^{-\eta_1} \int_{-\infty}^{x} u(z)e^{\eta_1 z} dz = 0. \tag{30}
\]
Note that now the OIDE (30) does not involve the function $g$. Moreover, if we consider the OIDE $(Lu)(x) = \alpha u(x)$ not for $x < x_0$ but for any $x \in \mathbb{R}$, the same procedure as above will lead to the same OIDE (30), since the only difference is replacing $\int_{x_0}^{x} u(z)e^{-\eta_1 z} dz + C_0$ in (28) and (29) by $\int_{x}^{+\infty} u(z)e^{-\eta_1 z} dz$, both of which lead to the same result after differentiating w.r.t. $x$. Consequently, without loss of generality, we can focus only on the OIDE (6) with a slight change, i.e., $(Lu)(x) = \alpha u(x)$ holds for any $x \in \mathbb{R}$.

A key point of the problem is how to deal with the integral part in the OIDE (6), which we rewrite as the following
\[
(IT) = \int_{-\infty}^{+\infty} u(x + y)f_Y(y)dy
\]
\[
= \sum_{i=1}^{m} p_i \int_{0}^{+\infty} u(x + y) e^{-\eta_i y} dy + \sum_{j=1}^{n} q_j \int_{-\infty}^{0} u(x + y) e^{\theta_j y} dy.
\]

A transformation $z = x + y$ yields
\[
\eta_i \int_{0}^{+\infty} u(x + y) e^{-\eta_i y} dy = -\eta_i e^{\eta_i x} \int_{x}^{+\infty} u(z)e^{-\eta_i z} dz,
\]
\[
\theta_j \int_{-\infty}^{0} u(x + y) e^{\theta_j y} dy = \theta_j e^{-\theta_j x} \int_{-\infty}^{x} u(z)e^{\theta_j z} dz.
\]

So (IT) can also be expressed as follows.
\[
(IT) = \sum_{i=1}^{m} p_i (-\eta_i) e^{\eta_i x} \int_{+\infty}^{x} u(z)e^{-\eta_i z} dz + \sum_{j=1}^{n} q_j \theta_j e^{-\theta_j x} \int_{-\infty}^{x} u(z)e^{\theta_j z} dz.
\]

Therefore the OIDE (6) is given by
\[
\frac{\sigma^2}{2}u''(x) + \mu u'(x) - (\lambda + \alpha)u(x)
\]
\[+\lambda \left( \sum_{i=1}^{m} p_i (-\eta_i) e^{\eta_i x} \int_{+\infty}^{x} u(z)e^{-\eta_i z} dz + \sum_{j=1}^{n} q_j \theta_j e^{-\theta_j x} \int_{-\infty}^{x} u(z)e^{\theta_j z} dz \right) = 0.
\]

For notation simplicity, we can rewrite the above OIDE by combining the two parts of jumping up and down together:
\[
\frac{\sigma^2}{2}u''(x) + \mu u'(x) - (\lambda + \alpha)u(x) + \lambda \sum_{i=1}^{M} r_i \rho_i e^{-\rho_i x} \int_{\pm\infty}^{x} u(z)e^{\rho_i z} dz = 0, \tag{31}
\]

A-3
where
\[ M = m + n, \quad r_i = p_iq_i \quad \text{and} \quad \rho_i = -\eta_i, \quad \text{for} \quad i = 1, \ldots, m, \]
\[ r_i = g_iq_i - m \quad \text{and} \quad \rho_i = \theta_i - m, \quad \text{for} \quad i = m + 1, \ldots, m + n, \]
and the lower limit of the integral \( \int_{x \pm \infty}^{x} u(z)e^{\rho_i z}dz \) is equal to \( +\infty \) when \( \rho_i < 0 \) and \( -\infty \) when \( \rho_i > 0 \). Similarly the equation \( G(x) - \alpha = 0 \) can be expressed as
\[ G(x) - \alpha = \frac{\sigma^2}{2}x^2 + \mu x - (\alpha + \lambda) + \lambda \sum_{i=1}^{M} \frac{r_i\rho_i}{\rho_i + x} = 0. \] (32)

To demonstrate our idea clearly, we consider a more general OIDE (33) by investigating its connection with the rational equation (34).

\[ \sum_{k=0}^{N} a_k u^{(k)}(x) + \sum_{i=1}^{M} \left( r_i\rho_i e^{-\rho_i x} \int_{x}^{\pm \infty} u(z)e^{\rho_i z}dz \right) = 0 \] (33)

and
\[ \sum_{k=0}^{N} a_k x^k + \sum_{i=1}^{M} \frac{r_i\rho_i}{\rho_i + x} = 0. \] (34)

**Definition B.1** The characteristic vector of the OIDE (33) is defined to be
\[ (N, M, a_0, a_1, \ldots, a_N, r_1, \ldots, r_M, \rho_1, \ldots, \rho_M), \]
where \( N \) and \( M \) represent the order of the differential equation part and the number of integral terms, respectively, \( a_0, a_1, \ldots, a_N \) correspond to the coefficients of the differential terms, and \( r_1, \ldots, r_M, \rho_1, \ldots, \rho_M \) are associated with the integral parts.

**Definition B.2** The characteristic vector of the equation (34) is defined to be
\[ (N, M, a_0, a_1, \ldots, a_N, r_1, \ldots, r_M, \rho_1, \ldots, \rho_M), \]
where \( N \) and \( M \) represent the order of the polynomial part and the number of fraction terms, respectively, \( a_0, a_1, \ldots, a_N \) correspond to the coefficients of the polynomial terms, and \( r_1, \ldots, r_M, \rho_1, \ldots, \rho_M \) are associated with the fraction parts.

Since the OIDE (33) and the equation (34) can be uniquely determined by their characteristic vectors, we can simply use their characteristic vectors to represent them.

We can see that the OIDE (31) and the equation (32) are special cases of (33) and (34), respectively. Furthermore, using the above definitions, we have shown that the characteristic vectors of the OIDE (31) and the equation (32) are identical and are given by
\[ (2, M, -\lambda - \alpha, \mu, \frac{\sigma^2}{2}, \lambda r_1, \ldots, \lambda r_M, \rho_1, \ldots, \rho_M). \]
In general, we define two operators: \( A \) and \( B \), which act on the OIDE (33) and the function (34) respectively to produce a new OIDE and a new function with the same types as before. Let us expatiate on the details of the two operators in turn.

The operation of the operator \( A \) on the OIDE (33) w.r.t. \( \rho_M \) includes the following two steps:

**Step 1:** Multiplying both sides of (33) by \( e^{\rho_M x} \) yields
\[
\sum_{k=0}^{N} a_k e^{\rho_M x} u^{(k)}(x) + r_M \rho_M \int_{\pm \infty}^{x} u(z) e^{\rho_M z} dz
\]
\[
+ \sum_{i=1}^{M-1} \left( r_i (\rho_M - \rho_i) \int_{\pm \infty}^{x} u(z) e^{\rho_i z} dz \right) = 0.
\]

**Step 2:** Taking derivative on (35) w.r.t. \( x \) and then multiplying both sides by \( e^{-\rho_M x} \) yields a new OIDE
\[
a_N u^{(N+1)}(x) + \sum_{k=1}^{N} (\rho_M a_k + a_{k-1}) u^{(k)}(x) + \left( \rho_M a_0 + \sum_{i=1}^{M} r_i \rho_i \right) u(x)
\]
\[
+ \sum_{i=1}^{M-1} \left( r_i (\rho_M - \rho_i) \rho_i e^{-\rho_i x} \int_{\pm \infty}^{x} u(z) e^{\rho_i z} dz \right) = 0.
\]

By performing the operator \( A \) on the OIDE (33) w.r.t. \( \rho_M \), we decrease the number of integrals of OIDE (33) by one, and increase the order of the differentiation by one as well. Moreover, the new OIDE (36) has a characteristic vector given by: \((N + 1, M - 1, \rho_M a_0 + \sum_{i=1}^{M} r_i \rho_i, \rho_M a_1 + a_0, \cdots, \rho_M a_N + a_{N-1}, a_N, r_1(\rho_M - \rho_1), \cdots, r_{M-1}(\rho_M - \rho_{M-1}), \rho_1, \cdots, \rho_{M-1})\).

To summarize, the operator \( A \) can be defined as follows:

**Definition B.3** An operator \( A \) acting on an OIDE (33) w.r.t. \( \rho_M \) produces another OIDE and is defined as:
\[
A_{\rho_M}(N, M, a_0, a_1, \cdots, a_N, r_1, \cdots, r_M, \rho_1, \cdots, \rho_M)
\]
\[
= (N + 1, M - 1, \rho_M a_0 + \sum_{i=1}^{M} r_i \rho_i, \rho_M a_1 + a_0, \cdots, \rho_M a_N + a_{N-1}, a_N, r_1(\rho_M - \rho_1), \cdots, r_{M-1}(\rho_M - \rho_{M-1}), \rho_1, \cdots, \rho_{M-1}),
\]
where the original OIDE and the new OIDE are simply represented by their respective characteristic vectors.
On the other hand, the operation of the operator $B$ on the equation (34) w.r.t. $\rho_M$ can be done by multiplying both sides of (34) by $\rho_M + x$ and the final equation is given by:

$$a_N x^{N+1} + \sum_{k=1}^{N} (\rho_M a_k + a_{k-1}) x^k + \left( \rho_M a_0 + \sum_{i=1}^{M} r_i \rho_i \right) + \sum_{i=1}^{M-1} r_i (\rho_M - \rho_i) \rho_i = 0.$$  \hspace{1cm} (37)

Thus, by performing the operator $B$ on the equation (34), we decrease the number of fractions of (34) by one, and increase the order of the polynomial by one as well. Moreover, the final equation (37) has the same form as before, and its characteristic vector is given by: $(N + 1, M - 1, \rho_M a_0 + \sum_{i=1}^{M} r_i \rho_i, \rho_M a_1 + a_0, \cdots, \rho_M a_N + a_{N-1}, a_N, r_1(\rho_M - \rho_1), \cdots, r_{M-1}(\rho_M - \rho_{M-1}), \rho_1, \cdots, \rho_{M-1})$, which is the same as that of the OIDE (36).

To summarize, the operator $B$ can be defined as follows:

**Definition B.4** An operator $B$ acting on an equation (34) w.r.t. $\rho_M$ produces a new equation and is defined as:

$$B_{\rho_M}(N, M, a_0, a_1, \cdots, a_N, r_1, \cdots, r_M, \rho_1, \cdots, \rho_M) = (N + 1, M - 1, \rho_M a_0 + \sum_{i=1}^{M} r_i \rho_i, \rho_M a_1 + a_0, \cdots, \rho_M a_N + a_{N-1}, a_N, r_1(\rho_M - \rho_1), \cdots, r_{M-1}(\rho_M - \rho_{M-1}), \rho_1, \cdots, \rho_{M-1}),$$

where the original equation and the new equation are simply represented by their respective characteristic vectors.

The description above implies that operators $A$ and $B$, acting on an OIDE (33) and an equation (34) w.r.t. $\rho_j$ for any $j = 1, 2, \cdots, M$ respectively, do not change the equivalence of their characteristic vectors. We state this result as the following lemma.

**Lemma B.5** Given an OIDE (33) and an equation (34), which have the same characteristic vectors, performing the operators $A$ and $B$ on (33) and (34) w.r.t. $\rho_j$ for any $j = 1, 2, \cdots, M$ respectively, yields a new OIDE with the number of differential terms increased by one and the number of integral terms decreased by one; and a new equation with the number of the polynomial terms increased by one and the number of fraction terms decreased by one. Moreover the new OIDE and the new equation still have the same characteristic vectors.

Using the above lemma, performing the operator $A$ and $B$ on (31) and (32) repeatedly for $M$ times changes the OIDE (31) to an $M + 2$ order homogeneous linear ODE with constant coefficients, and changes the equation (32) to an $M + 2$ order polynomial equation, which is
exactly \((G(x) - \alpha)(\rho_1 + x) \cdots (\rho_M + x) = 0\). Lemma B.5 tells us that the final OIDE, actually an ODE, should have the same characteristic vectors as the final equation. Therefore, we conclude that the equation \((G(x) - \alpha)(\rho_1 + x) \cdots (\rho_M + x) = 0\) is actually the characteristic function of the \(M + 2\) order homogeneous linear ODE with constant coefficients, and Theorem 3.2 is proved. \(\square\)

C Proof of Theorem 3.3

Proof. For notation simplicity, we use \(\beta_1, \ldots, \beta_{m+1}, \gamma_1, \ldots, \gamma_{n+1}\) to represent \(\beta_{1,\alpha}, \ldots, \beta_{m+1,\alpha}, \gamma_{1,\alpha}, \ldots, \gamma_{n+1,\alpha}\), which are the \((m + n + 2)\) roots of the equation \(G(x) = \alpha\) and satisfy (7).

If \(x \geq b\), \(Ex[e^{-\alpha \tau b + \theta X_0}] = e^{\theta x}\) because \(\tau b = 0\) and \(X_0 = x\). When \(x < b\), \(Ex[e^{-\alpha \tau b + \theta X_0}]\), as a function of \(x\), is closely related to the solution of the OIDE \((Lu)(x) = \alpha u(x)\). If we can solve this OIDE explicitly, then we can prove this theorem by using a similar technique as in [35]. Therefore, we intend to prove Theorem 3.3 in the following three steps.

(I). Prove that the OIDE \((Lu)(x) = \alpha u(x)\) when \(x < b\) and \(u(x) = e^{\theta x}\) when \(x \geq b\) has a unique solution given that \(u(x)\) satisfies the following two conditions: (i) the smooth pasting condition, i.e., \(u(b-) = u(b+) = e^{\theta b}\) and (ii) the bounded boundary condition, i.e., \(u(x)\) is bounded near \(-\infty\). Furthermore, the solution can be expressed explicitly as \(u(x) = \sum_{i=1}^{m+1} w_i e^{\beta_i x}\) when \(x < b\), where \(w_i\) is the solution of the linear system \(ABw = J\). Here \(A, B\) and \(J\) are the same as in Theorem 3.3.

(II). Prove that the following function

\[
u(x) := \begin{cases} e^{\theta x} & \text{if } x \geq b \\
\sum_{i=1}^{m+1} w_i e^{\beta_i x} & \text{if } x < b\end{cases}
\]

is exactly \(Ex[e^{-\alpha \tau b + \theta X_0}]\).

(III). The matrix \(A\) is nonsingular.

Proof of (I): Applying Theorem 3.2 yields that when \(x < b\), \(u(x)\) is of the following form

\[u(x) = \sum_{i=1}^{m+1} w_i e^{\beta_i x} + \sum_{j=1}^{n+1} v_j e^{\gamma_j x},\]

where \(w_1, w_2, \ldots, w_{m+1}, v_1, v_2, \ldots, v_{n+1}\) are undetermined constants. Since \(u(x)\) is bounded near \(-\infty\), we conclude that

\[v_1 = v_2 = \cdots = v_{n+1} = 0.\]

On the other hand, the smooth pasting condition implies that

\[w_1 e^{\beta_1 b} + w_2 e^{\beta_2 b} + \cdots + w_{m+1} e^{\beta_{m+1} b} = e^{\theta b}.\]
It is easy to see that to determine $w_1, w_2, \cdots, w_{m+1}, m$ more equations are required. To achieve this, we substitute $u(x) = \sum_{l=1}^{m+1} w_l e^{\beta_l x}$ back into the original OIDE. Note that for any $x < b$ and $\theta < \eta_1$,

\[
\int_{-\infty}^{+\infty} u(x+y) f_Y(y) dy = \int_{-\infty}^{0} u(x+y) \left( q_d \sum_{j=1}^{n} q_j \frac{\theta_j}{e^{\theta_j y}} \right) dy + \int_{0}^{b-x} u(x+y) \left( p_u \sum_{i=1}^{m} p_i \eta_i e^{-\eta_i y} \right) dy + \int_{b-x}^{+\infty} e^{\theta(x+y)} \left( p_u \sum_{i=1}^{m} p_i \eta_i e^{-\eta_i y} \right) dy
\]

\[
= q_d \sum_{j=1}^{n} \left( q_j \frac{\theta_j}{e^{\theta_j x}} \int_{-\infty}^{x} u(z) e^{\theta_j z} dz \right) + p_u \sum_{i=1}^{m} \left( p_i \eta_i e^{\eta_i x} \int_{x}^{b} u(z) e^{-\eta_i z} dz \right)
\]

\[
+ \int_{b-x}^{+\infty} e^{\theta x} \left( p_u \sum_{i=1}^{m} p_i \eta_i e^{-(\eta_i-\theta)y} \right) dy
\]

\[
= q_d \sum_{j=1}^{n} \sum_{l=1}^{m+1} \left( w_l \frac{\theta_j}{\theta_j + \beta_l} e^{\beta_l x} \right) + p_u \sum_{i=1}^{m} \sum_{l=1}^{m+1} \left( w_l \frac{p_i \eta_i}{\eta_i - \beta_l} e^{\beta_l x} \right)
\]

\[
- p_u \sum_{i=1}^{m} \sum_{l=1}^{m+1} \left( w_l \frac{p_i \eta_i}{\eta_i - \beta_l} e^{\beta_l e^{\eta_i(x-b)}} \right) + p_u \sum_{i=1}^{m} \left( \frac{p_i \eta_i}{\eta_i - \theta} e^{\theta x} e^{\eta_i(x-b)} \right).
\]

So for any $x < b$ and $\theta < \eta_1$, we have:

\[
\frac{\sigma^2}{2} u''(x) + \mu u'(x) - (\lambda + \alpha) u(x) + \lambda \int_{-\infty}^{+\infty} u(x+y) f_Y(y) dy
\]

\[
= \sum_{l=1}^{m+1} \left[ w_l e^{\beta_l x} (G(\beta_l) - \alpha) \right] - \lambda p_u \sum_{i=1}^{m} \left\{ p_i e^{\eta_i(x-b)} \left[ \sum_{l=1}^{m+1} \left( w_l \frac{\eta_i e^{\beta_l}}{\eta_i - \beta_l} \right) - \frac{\eta_i}{\eta_i - \theta} e^{\theta b} \right] \right\}
\]

\[
= -\lambda p_u \sum_{i=1}^{m} \left\{ p_i e^{\eta_i(x-b)} \left[ \sum_{l=1}^{m+1} \left( w_l \frac{\eta_i e^{\beta_l}}{\eta_i - \beta_l} \right) - \frac{\eta_i}{\eta_i - \theta} e^{\theta b} \right] \right\}
\]

\[
= 0.
\]

Because $\eta_1, \eta_2, \cdots, \eta_m$ are distinct, we conclude that:

\[
\sum_{l=1}^{m+1} \left( \frac{\eta_i e^{\beta_l}}{\eta_i - \beta_l} w_l \right) = \frac{\eta_i}{\eta_i - \theta} e^{\theta b} \quad \text{for} \quad i = 1, 2, \cdots, m
\]  \hspace{1cm} (39)

Combining (38) and (39) results in $ABw = J$ immediately.

**Proof of (II):** Consider the following function

\[
u(x) = \begin{cases} 
  e^{\theta x} & \text{if } x \geq b \\
  \sum_{l=1}^{m+1} w_l e^{\beta_l x} & \text{if } x < b.
\end{cases}
\]

A-8
Now we begin to prove this function $u(x)$ is exactly $E^{x}[e^{-\alpha \tau_b + \theta X_b}]$.

First, from Part (I), we know

$$-\alpha u(x) + (Lu)(x) = 0, \quad \text{for any } x < b. \quad (40)$$

Since $u(x)$ may not be continuously differentiable (e.g., when $m = n = 1$, i.e., for the DEP, Kou and Wang [35] pointed out that if $\theta = 0$, then $u'(b-) > 0 = u'(b+)$), we cannot apply Itô’s formula to the process $\{e^{-\alpha t}u(X_t) : t \geq 0\}$ directly. However, we can construct a series of functions $\{u_n(x) : n = 1, 2, \cdots\}$ smooth enough to approximate $u(x)$. More precisely, $\{u_n(x) : n = 1, 2, \cdots\}$ can be selected such that: (1) $u_n(x)$ converges to $u(x)$ as $n$ goes to $+\infty$ for any $x$; (2) $u_n(x)$ is twice continuously differentiable for any $n = 1, 2, \cdots$; (3) $u_n(x) \equiv u(x)$ for any $x \leq b$ or $x > b + \frac{1}{n}$; (4) for all $x \in (b, b + \frac{1}{n})$ and $n$, $0 \leq u_n(x) \leq M_1$, where $M_1$ is a positive constant. In addition, for all $x \leq b$, we have $|u_n(x)| \equiv |u(x)| \leq M_2$ for any $n = 1, 2, \cdots$, where $M_2 := \sum_{i=1}^{m+1} |w_i| e^{\beta b}$.

Similar algebra as on p. 510 of Kou and Wang [35] yields that for any $x < b$,

$$(Lu_n)(x) = \alpha u(x) + \lambda \int_{b-x}^{b-x+1/n} [u_n(x+y) - u(x+y)] f(y) dy,$$

thanks to (40). Note that by construction,

$$|u_n(x) - u(x)| \leq \max_{x \in (b, b+1/n)} u_n(x) + \max_{x \in (b, b+1/n)} u(x) \\
\leq M_1 + \max(e^{\beta b}, e^{\theta(b+1)}) =: M \quad \text{for any } x \text{ and } n.$$

We obtain that

$$| -\alpha u_n(x) + (Lu_n)(x) | \\
\leq \lambda \rho u \sum_{i=1}^{m} |p_i| \eta_i \int_{b-x}^{b-x+1/n} |u_n(x+y) - u(x+y)| dy \\
\leq \lambda \rho u M \sum_{i=1}^{m} |p_i| \eta_i \to 0, \quad \text{uniformly for all } x < b, \text{ as } n \to +\infty.$$

Applying Itô’s formula to the jump processes $\{e^{-\alpha t}u_n(X_t) : t \geq 0\}$ for any $n = 1, 2, \cdots$, we obtain a series of local martingales $\{M_t^{(n)} : t \geq 0\}$ for $n = 1, 2, \cdots$, as follows:

$$M_t^{(n)} := e^{-\alpha t \wedge \tau_b} u_n(X_{t \wedge \tau_b}) - \int_{0}^{t \wedge \tau_b} e^{-\alpha s} [-\alpha u_n(X_s) + (Lu_n)(X_s)] ds.$$

Note that for any $t \geq 0$ and $n = 1, 2, \cdots$,

$$|u_n(X_{t \wedge \tau_b})| \\
= \left| u_n(X_{t \wedge \tau_b}) I_{\{t < \tau_b\}} + u_n(X_{t \wedge \tau_b}) I_{\{t \geq \tau_b, X_{\tau_b} < b + 1/n\}} + u_n(X_{t \wedge \tau_b}) I_{\{t \geq \tau_b, X_{\tau_b} > b + 1/n\}} \right| \\
\leq \max(M_2, M_1, e^{\theta X_{\tau_b}} I_{\{t \geq \tau_b\}}), \quad (42)$$
where the inequality holds because $|u_n(x)|$ is bounded by $M_2$ when $x \leq b$, bounded by $M_1$ when $x \in (b, b + 1/n)$, and equal to $u(x) \equiv e^{\theta x}$ when $x \geq b + 1/n$. Hence, for any $t \geq 0$ and $n = 1, 2, \ldots$, we have

$$|M_t^{(n)}| \leq |u_n(X_{t \wedge \tau_n})| + \int_0^{t \wedge \tau_n} |-\alpha u_n(X_s) + (Lu_n)(X_s)| \, ds$$

$$\leq \max(M_1, M_2, e^{\theta X_{\tau_n}} I_{\{t \geq \tau_n\}}) + \frac{\lambda p_u M \sum_{i=1}^n |p_i| \eta_i t}{n}.$$  (43)

If $\theta \leq 0$, then

$$\sup_{0 \leq s \leq t} |M_s^{(n)}| \leq \max(M_1, M_2, e^{\theta b}) + \frac{\lambda p_u M \sum_{i=1}^n |p_i| \eta_i t}{n},$$

which implies that $\{M_t^{(n)} : t \geq 0\}$ is a true martingale for any $n = 1, 2, \ldots$.

If $\theta \in (0, \eta_1)$, we have

$$\sup_{0 \leq s \leq t} |M_s^{(n)}| \leq \max(M_1, M_2, e^{\theta \sup_{0 \leq s \leq t} X_s}) + \frac{\lambda p_u M \sum_{i=1}^m |p_i| \eta_i t}{n},$$

In order to show that in this case, $\{M_t^{(n)} : t \geq 0\}$ is also a true martingale for any $n = 1, 2, \ldots$, it suffices to prove that

$$E^x \left[e^{\theta \sup_{0 \leq s \leq t} X_s} \right] < +\infty, \quad \text{for any } t \geq 0.$$

Note that

$$\sup_{0 \leq s \leq t} X_s \leq X_0 + |\mu| t + \sigma \max_{0 \leq s \leq t} W_s + \sum_{i=1}^{N_t} Y_i^+,$$

where $Y_i^+ := \max\{Y_i, 0\}$, and $\max_{0 \leq s \leq t} W_s$ has the same distribution as $|W_t|$. It follows that for any $t \geq 0$ and $\theta \in (0, \eta_1),$

$$E^x \left[\left.e^{\theta \sup_{0 \leq s \leq t} X_s} \right| X_0 \right] \leq e^{\theta(|x| + |\mu| t)} \cdot E \left[e^{\theta \sigma |W_1|} \right] \cdot E \left[e^{\theta \sum_{i=1}^{N_t} Y_i^+} \right] < +\infty$$  (44)

because on the one hand, the fact that $\theta < \eta_1$ leads to

$$E \left[e^{\theta \sum_{i=1}^{N_t} Y_i^+} \right] = e^{\lambda t (E e^{\beta Y_1^+} - 1)} = e^{\lambda (q_0 + p_n \sum_{i=1}^{n} \frac{|Y_i|}{n} - 1)} < +\infty,$$

and on the other hand, $E \left[e^{\theta \sigma |W_1|} \right] = 2e^{\theta^2 \sigma^2/2} \Phi(\theta \sigma \sqrt{7}) < +\infty$, where $\Phi(\cdot)$ is the cdf of the standard normal random variable.

Therefore, we conclude that for any $\theta < \eta_1$, $\{M_t^{(n)} : t \geq 0\}$ is a true martingale for any $n = 1, 2, \ldots$. Thus, for any $t \geq 0$ and $x < b$, we have

$$E^x M_t^{(n)} = E^x \left[\left.e^{-\alpha(t \wedge \tau_n)} u_n(X_{t \wedge \tau_n}) \right| X_0 \right] - E^x \left[\left.\int_0^{t \wedge \tau_n} e^{-\alpha s} [-\alpha u_n(X_s) + (Lu_n)(X_s)] \, ds \right| X_0 \right]$$

$$= E^x M_t^{(n)} = u_n(X_0) = u_n(x) = u(x),$$

A-10
where \( x = X_0 \) is the starting point of \( \{ X_t : t \geq 0 \} \). Letting \( n \) go to \( +\infty \) and applying the dominated convergence theorem (DCT) yields that

\[
\lim_{n \to +\infty} E^x \left[ e^{-\alpha(t \land \tau_n)} u_n(X_{t \land \tau_n}) \right] = E^x \left[ e^{-\alpha(t \land \tau_n)} u(X_{t \land \tau_n}) \right],
\]

(45)

and

\[
\lim_{n \to +\infty} E^x \left[ \int_0^{t \land \tau_n} e^{-\alpha s} \left[ -\alpha u_n(X_s) + (Lu_n)(X_s) \right] \, ds \right] = 0,
\]

(46)

where the DCT applies for (45) because (42) and (46) holds thanks to the uniform convergence in (41). Consequently, we obtain that for any \( t \geq 0 \) and \( x < b \),

\[
u(x) = \begin{aligned}
E^x \left[ e^{-\alpha(t \land \tau_n)} u(X_{t \land \tau_n}) \right] &= E^x \left[ e^{-\alpha t} u(X_t) I_{\{ \tau_n \leq t \}} \right] + E^x \left[ e^{-\alpha t} u(X_t) I_{\{ \tau_n > t \}} \right], \\
&= E^x \left[ e^{-\alpha t} u(X_t) I_{\{ \tau_n \leq t \}} \right] + E^x \left[ e^{-\alpha t} u(X_t) I_{\{ \tau_n > t \}} \right],
\end{aligned}
\]

(47)

where the last equality holds because \( u(X_\tau) = e^{\theta X_\tau} \) on the set \( \{ \tau_b < +\infty \} \).

Note that on the set \( \{ \tau_b > t \} \), we have \( X_t < b \) and hence \( |u(X_t)| \leq M_2 \). It follows from the DCT that the second term on the right-hand side of (47) converges to zero as \( t \) goes to infinity. Besides, applying the monotone convergence theorem yields that the first term on the right-hand side of (47) converges to \( E^x \left[ e^{-\alpha \tau_b + \theta X_{\tau_b}} I_{\{ \tau_b < +\infty \}} \right] \) as \( t \) goes to infinity.

Consequently, letting \( t \) go to \( +\infty \) in (47) yields

\[
u(x) = \begin{aligned}
E^x \left[ e^{-\alpha \tau_b + \theta X_{\tau_b}} I_{\{ \tau_b < +\infty \}} \right] &= E^x \left[ e^{-\alpha \tau_b + \theta X_{\tau_b}} \right], \quad \text{for any } x < b.
\end{aligned}
\]

Proof of (III). We will prove it by contradiction. Assume that \( A \) is singular. Then the \((m+1)\) row vectors of \( A \) are linearly dependant. Thus, there exist \((m+1)\) constants \( k_0, k_1, \cdots, k_m \), not all of which are zero, such that

\[
k_0 + k_1 \frac{\eta_1}{\eta_1 - \beta_1} + k_2 \frac{\eta_2}{\eta_2 - \beta_1} + \cdots + k_m \frac{\eta_m}{\eta_m - \beta_1} = 0, \quad \text{for } l = 1, 2, \cdots, m + 1.
\]

It implies that the function \( f_A(\beta) := k_0 + k_1 \frac{\eta_1}{\eta_1 - \beta} + k_2 \frac{\eta_2}{\eta_2 - \beta} + \cdots + k_m \frac{\eta_m}{\eta_m - \beta} \) has at least \((m+1)\) roots: \( \beta_1, \beta_2, \cdots, \beta_{m+1} \). However, \( f_A(\beta) \prod_{i=1}^{m} \left( (\eta_i - \beta) I_{\{k_i \neq 0\}} \right) \) is a polynomial with an order at most \( m \), and therefore has at most \( m \) roots. Thus, \( f_A(\beta) \) also has at most \( m \) roots, which renders a contradiction. Therefore, the matrix \( A \) is non-singular.
D Proof of Theorem 3.4

Proof. Note that for $\theta > 0$,

$$
\mathcal{L}(\alpha, \theta) = \int_0^{+\infty} \int_{-\infty}^{+\infty} e^{-\theta \hat{a} - \alpha t} E^0 [I_{\{X_t \geq -\hat{a}, \tau_b \leq t\}}] \hat{a} \, dt
$$

$$
= E^0 \left\{ \int_{\tau_b}^{+\infty} \left[ \int_{-\infty}^{+\infty} e^{-\theta \hat{a} - \alpha t} \, d\hat{a} \right] \, dt \right\}
$$

$$
= \frac{1}{\theta} E^0 \left\{ \int_{\tau_b}^{+\infty} e^{\theta X_t - \alpha t} \, dt \right\}
$$

$$
= \frac{1}{\theta} E^0 \left[ e^{-\alpha \tau_b} \int_0^{+\infty} e^{\theta X_t + \tau_b - \alpha t} \, dt \right].
$$

On the other hand, the strong Markov property implies that for any $\alpha > \max(G(\theta), 0)$,

$$
E^0 \left[ e^{-\alpha \tau_b} \int_0^{+\infty} e^{\theta X_t + \tau_b - \alpha t} \, dt \bigg| \mathcal{F}_{\tau_b} \right]
$$

$$
= e^{-\alpha \tau_b + \theta X_{\tau_b}} E^0 \left[ \int_0^{+\infty} e^{\theta X_t - \alpha t} \, dt \right]
$$

$$
= e^{-\alpha \tau_b + \theta X_{\tau_b}} e^{(G(\theta) - \alpha) \tau_b} dt = \frac{e^{-\alpha \tau_b + \theta X_{\tau_b}}}{\alpha - G(\theta)}.
$$

Combining them together and applying (9) yields (16) immediately. □

E Distribution of the Running Maxima

This section gives the closed-form Laplace transform of the running maxima of the process $X_t$, i.e. $M_X(t) := \max_{0 \leq s \leq t} X_s$, where $X_0 = 0$.

**Theorem E.1** Denote by $\mathcal{L}_M(s)$ the Laplace transforms of $E^0[e^{v M_X(t)}]$ w.r.t. $t$ evaluated at sufficiently large $s > 0$. More precisely, $\mathcal{L}_M(s) = \int_0^{+\infty} e^{-st} E^0[e^{v M_X(t)}] \, dt$, Then for any $v \in (-\infty, \beta_1, s)$, we have

$$
\mathcal{L}_M(s) = \frac{1}{s} + \frac{v}{s} \sum_{l=1}^{m+1} \frac{d_l}{\beta_{l,s} - v}, \quad s > 0,
$$

(48)

where $\beta_1, \beta_2, \cdots, \beta_{m+1}$ are the $(m+1)$ positive roots of the equation $G(x) = s$ in (3) and $d = (d_1, \cdots, d_{m+1})'$ is uniquely determined by $Ad = 1$, where $A$ associated with $s$ is defined in Theorem 3.3 and $1 = (1, 1, \cdots, 1)'$.

Before proving Theorem E.1, we present the following Lemma E.2.
Lemma E.2 Assume that $v \in (-\infty, \beta_{1,s})$, where $\beta_{1,s}$ is the smallest positive root of the equation $G(x) = s$ for sufficiently large $s > 0$. Then for any $t > 0$, we have that

$$\lim_{y \to +\infty} e^{vy}P[M_X(t) \geq y] = 0 \quad (49)$$

Proof. When $v \leq 0$, the conclusion is trivial. When $v \in (0, \beta_{1,s})$, note that the process \{\sigma_t \geq 0\} is a martingale for any $\theta \in (-\theta_1, \eta_1)$ since $G(\theta)$ is the exponent of the Lévy process $\{X_t : t \geq 0\}$. Fix $\theta \in (v, \beta_{1,s})$ such that $G(\theta) > 0$. This $\theta$ must exist because $G(\beta_{1,s}) = s > 0$ and $G(\theta)$ is continuous in the interval $(v, \beta_{1,s})$. Note that $e^{\theta y}P[\tau_y \leq t] \leq E[e^{\theta X_t \wedge \tau_y}] \leq e^{G(\theta)t}E[e^{\theta X_t \wedge \tau_y - G(\theta)(t \wedge \tau_y)}] \leq e^{G(\theta)t}$, where the last equality holds owing to optional sampling theorem. So for any $y > 0$, we have $e^{vy}P[M_X(t) \geq y] = e^{(v-\theta)y}e^{\theta y}P[M_X(t) \geq y] = e^{(v-\theta)y}e^{\theta y}P[\tau_y \leq t] \leq e^{(v-\theta)y}e^{G(\theta)t}$.

Note that $\theta > v$, so letting $y$ go to infinity completes the proof of (49). $\square$

Now we are ready to prove Theorem E.1.

Proof. First, using integration by parts and applying (49) leads to

$$E e^{vM_X(t)} = \int_0^\infty e^{vy}f_M(y)dy = -\int_0^\infty e^{vy}dP(M_X(t) \geq y)$$

$$= 1 + v\int_0^\infty P(M_X(t) \geq y)e^{vy}dy,$$

where $f_M(y)$ represents the pdf of $M_X(t)$. Accordingly, $L_M(s)$ can be expressed as

$$L_M(s) = \int_0^\infty e^{-st}E[e^{vM_X(t)}]dt$$

$$= \frac{1}{s} + v\int_0^\infty e^{-st}\left[\int_0^\infty e^{vy}P(M_X(t) \geq y)dy\right]dt$$

$$= \frac{1}{s} + v\int_0^\infty e^{vy}\left[\int_0^\infty e^{-st}P(M_X(t) \geq y)dt\right]dy.$$

On the other hand, we have

$$\int_0^\infty e^{-st}P(M_X(t) \geq y)dt$$

$$= -\frac{1}{s} \int_0^\infty P(M_X(t) \geq y)e^{-st}dt$$

$$= \frac{1}{s} \int_0^\infty e^{-st}dP(M_X(t) \geq y)$$

$$= \frac{1}{s} \int_0^\infty e^{-st}dP(\tau_y \leq t) = \frac{1}{s} E e^{-s\tau_y}.$$
Thus, by (13) with $x = 0$,

\[
\mathcal{L}_M(s) = \frac{1}{s} + v \int_0^\infty e^{vy} \frac{1}{s} e^{-sy} dy
\]

\[
= \frac{1}{s} + \frac{v}{s} \sum_{l=1}^{m+1} d_l \int_0^\infty e^{vy} e^{-\beta_l s y} dy = \frac{1}{s} + \frac{v}{s} \sum_{l=1}^{m+1} \frac{d_l}{\beta_l s - v},
\]

from which the conclusion follows. □