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Fourier transformation and the pricing of average-rate derivatives

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Abstract In this article we propose a method to compute the density of the arithmetic average of a Markov process. This approach is then applied to the pricing of average rate options (Asian options). It is demonstrated that as long as a closed form formula is available for the discount bond price when the underlying process is treated as the riskless interest rate, analytical formulas for the density function of the arithmetic average and the Asian option price can be derived. This includes the affine class of term structure models. The Cox et al. (1985) square root interest rate process is used as an example. When the underlying process follows a geometric Brownian motion, a very efficient numerical method is proposed for computing the density function of the average. Extensions of the techniques to the cases of multiple state variables are also discussed.

Keywords Fourier transformation · Average-rate derivaties · Forward risk-neutral measure

JEL Classification G13

Although Asian options are not traded on any exchanges, they are among the most actively traded derivatives in the over-the-counter markets. There are several reasons that they are so popular. First, because the payoff function of an Asian option not only depends on the value of the underlying process at the maturity date but also depends

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on the whole sample path for some finite interval, price manipulation is less likely and has less impact on the payoff of the option. Second, since the payoff function depends on the average of the underlying process, it also reduces the risk associated with the value of the underlying process on the maturity date. Third, if the objective is to hedge against average movements of the underlying asset, an average-rate option is obviously a more appropriate instrument to use than regular options.

Features of Asian options appear in many different types of contracts. For example, options on oils are typically settled against an average of daily or monthly prices. In the foreign exchange market, Asian options can be used to hedge against currency movements. Similarly, Asian options on interest rate can be used to hedge against interest rate movements. Another type is the catastrophe loss option analyzed in Bakshi and Madan (2002).

Asian options belong to the so-called path-dependent derivatives. Path-dependent derivatives are among the most difficult derivatives to price and hedge both analytically and numerically. Several approaches have been proposed in the literature to price and hedge Asian options. Kemna and Vorst (1990) use Monte Carlo simulation to price and hedge Asian options. To reduce the variance of their simulation, they use the corresponding geometric average option as their control variable. Although Monte Carlo simulation is a very flexible method for pricing path-dependent European options, it is very time-consuming. Their method also has the drawback that if the underlying process for averaging is not log-normal, a closed form solution for some control variable may not exist. Therefore in order to ensure convergence, even longer time-consuming simulation may be needed. A further drawback of their approach is that to compute the hedging parameters, a new simulation is needed starting with a slightly different initial stock price. Obviously, less time-consuming methods are desirable.

Rogers and Shi (1995) succeed in advancing the pricing of Asian options in another direction. By making a suitable change of variables, they are able to transform the original three-dimensional (including the time dimension) partial differential equation (PDE) satisfied by the price of an Asian option into a two-dimensional PDE.[1] This is a tremendous reduction of complexity in terms of computation. But their approach requires the underlying process to be log-normal and the terminal payoff to be simple functions of the average rate, like those of regular Asian calls and puts. Furthermore, their PDE has to be solved for each different option.

Turnbull and Wakeman (1991) use an Edgeworth series expansion to approximate the density function of the average rate. They succeed in deriving closed form formulas for the Asian options. Their approach, however, suffers two drawbacks. First, convergence is not assured for all possible parameters; Second, their method does not apply if the underlying process is not log-normal. Levy (1992) uses the log-normal density as a first-order approximation to the true density of the average price. Levy's approximation corresponds to that of Turnbull and Wakeman by keeping only the first term in the Edgeworth series expansion and thus shares the same drawbacks as those

[1] In the context of floating strike Asian options, Ingersoll (1987) also succeeded in reducing the dimension of PDE.

of Turnbull and Wakeman's approach. Another analytical approximation for Asian and basket options on log-normal processes is Ju (2002).

One class of semi-analytical methods is based on a transformation of the underlying state variable whose direct characterization may be difficulty, e.g., the characteristic function (or relatedly the Fourier transformation or Laplace transformation) of the underlying random variable. This approach has been applied to greater success in Heston (1993), Geman and Yor (1993), Bakshi and Madan (2000), and Duffie et al. (2000). Our article belongs to this genre. Even though it is very difficult to determine the density function of the average value of a general Markov process, it will be shown in Sect. 1 that the Fourier transformation of the density function satisfies a PDE with a simple terminal condition. The Fourier inverse transformation yields the density function. The price of an Asian option is then simply the expectation of the discounted terminal payoff.

It is further shown that the computation can be simplified tremendously for two of the most widely used stochastic processes in finance, the Cox et al. (1985) square-root process for interest rates and the log-normal process for equities. In the former case, closed form formulas[2] are obtained for the density function of the average interest rate and the prices of Asian options. In the latter case, a very efficient numerical procedure is proposed to compute the density function of the average equity prices.

For a general Markov process, the density function can be obtained in a straightforward manner at considerable more computing cost. But this method could still be preferred. First, one still only needs to solve a PDE with a lower dimensionality. Second, once the density function is obtained, fast numerical integration can be used to price all average price options. These options could be regular Asian calls and puts with different strikes or some other Asian derivatives whose payoffs can be any functional form of the average price. On the other hand, the PDE-based methods would require solving the PDE once for each different option. Lastly, hedging parameters can be obtained with minimal additional computing cost.

The remainder of the article is organized as follows. A general approach for computing the density function of the average value of a general Markov process is proposed in Sect. 1. In Sect. 2, the method developed in Sect. 1 is applied to the Cox et al. (1985) square-root process for pricing average interest rate options. The approach is specialized to the familiar log-normal process for equity Asian options in Sect. 3. Section 4 discusses extensions of the techniques to multiple state variables models, including multi-factor interest rate models for pricing average interest rate options and stochastic interest rate models for pricing equity average rate options. Section 5 summarizes and concludes the article. The more technical details, including the change of probability measure and its application to the Cox et al. (1985) square-root process, are provided in Appendices.

[2] It is in the same sense that the normal distribution function is considered closed form even though it is represented by an integral.

1 Computing the density function of the average-rate process

In this section the general partial differential equation (PDE) satisfied by the characteristic function of the arithmetic sum[3] during a finite interval for a general Markov process is derived. A connection between the Fourier transformation of the density function of the sum[4] and the characteristic function of the sum is established.[5] This observation allows one to obtain the density function of the sum by inverting the Fourier transformation.

Consider a nonnegative Markov process:

$$\mathrm{d}x(t) = \mu(x,t)\mathrm{d}t + \sigma(x,t)\mathrm{d}w(t), \tag{1}$$

where $w(t)$ is a standard Wiener process. Let $y(t) = \int_0^t x(u)\mathrm{d}u$ represent the arithmetic sum of $x(\cdot)$. The characteristic function of $y(t)$ is defined by

$$F(x(0),\eta,t) = E_0\left[\exp(-i\eta y)\right] = \int_0^{\infty} e^{-i\eta y} f(y)\mathrm{d}y, \tag{2}$$

where $f(\cdot)$ is the density function of $y(t)$.

Next, a new function $g(\cdot)$ is defined by

$$g(y) = \begin{cases} f(y), & \text{if } y \geq 0 \\ f(-y), & \text{if } y < 0. \end{cases}$$

That is, $g(\cdot)$ is $f(\cdot)$ extended evenly to the negative axis.[6] The Fourier transformation of $g(\cdot)$ is defined by

$$G(\eta) = \int_{-\infty}^{\infty} e^{-i\eta y} g(y)\mathrm{d}y. \tag{3}$$

Since $g(\cdot)$ is real and even, it is easy to see that

$$G(\eta) = \int_0^{\infty} 2\cos(\eta y) g(y)\mathrm{d}y = 2R\left(\int_0^{\infty} e^{-i\eta y} f(y)\mathrm{d}y\right) = 2R(F(x(0),\eta,t)) \tag{4}$$

is real and even too. $R(F(x(0),\eta,t))$ denotes the real part of $F(x(0),\eta,t)$.

[3] For ease of exposition, the arithmetic sum instead of the arithmetic average is used. The density functions of the two relate to each other by the scaling of the length of the averaging interval. For pricing Asian options, one can scale the strike price and then use the density function of the arithmetic sum.

[4] "Sum" refers to the arithmetic sum (integration) of the values of the underlying process during a finite interval.

[5] In this article, it is assumed that the underlying Markov process is nonnegative because most of the stochastic processes in finance have this property. If the underlying process can have any real values, the characteristic function of the sum *is* the Fourier transformation of the density function of the sum. The method developed here would apply with little change.

[6] One could have assumed $g(\cdot)$ to be zero along the negative axis, but then the Fourier transformation will be complex. The present extension of the function is preferred.

The inverse transformation is given by (only need to consider $y \geq 0$)

$$g(y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\eta y} G(\eta) \mathrm{d}\eta. \tag{5}$$

Since $G(\cdot)$ is real and even,

$$g(y) = \frac{1}{2\pi} \int_{0}^{\infty} 2\cos(\eta y) G(\eta) \mathrm{d}\eta. \tag{6}$$

Therefore,

$$f(y) = \frac{2}{\pi} \int_{0}^{\infty} \cos(\eta y) R(F(x(0), \eta, t)), \quad y \geq 0. \tag{7}$$

Consequently, if $F(x(0), \eta, t)$ is found, the Fourier inverse transformation can be used to compute the density function of the sum.

To find $F(x(0), \eta, t)$, define

$$F(x(s), \eta, t-s) = E_s \left[\exp\left(-i\eta \int_{s}^{t} x(u) \mathrm{d}u \right) \right]. \tag{8}$$

It is clear that $F(x(s), \eta, t-s)$ is similar to the bond price at time s with maturity date t for an imaginary interest rate process $i\eta x(u)$. As a matter of fact, $F(x(s), \eta, t-s)$ satisfies a PDE similar to the one satisfied by the bond price if $x(t)$ is the interest rate under the risk-neutral measure. Formally, define

$$\begin{aligned} F^*(x(s), \eta, t-s) &= \exp\left(-i\eta \int_{0}^{s} x(u) \mathrm{d}u \right) F(x(s), \eta, t-s) \\ &= E_s \left[\exp(-i\eta \int_{0}^{t} x(u) \mathrm{d}u) \right]. \end{aligned} \tag{9}$$

Since the argument of the conditional expectation on the right-hand side is independent of s, the left-hand side is a martingale. Therefore the drift of $F^*(x(s), \eta, t-s)$ must vanish. Ito's lemma yields

$$\begin{aligned} \mathrm{d}F^* = {} & \exp\left(-i\eta \int_{0}^{s} x(u) \mathrm{d}u \right) \left[-i\eta x F + F_s + \mu(x, s) F_x + \frac{1}{2}\sigma^2(x, s) F_{xx} \right] \mathrm{d}s \\ & + \exp\left(-i\eta \int_{0}^{s} x(u) \mathrm{d}u \right) \sigma(x, s) F_x \mathrm{d}w. \end{aligned} \tag{10}$$

Setting the drift of $\mathrm{d}F^*$ equal to zero, the following PDE

$$- i\eta x F + F_s + \mu(x, s) F_x + \frac{1}{2}\sigma^2(x, s) F_{xx} = 0 \tag{11}$$

is obtained. The terminal condition is

$$F(x(t), \eta, 0) = (1, 0).^7$$

As just mentioned, this PDE is the fundamental valuation equation for any contingent claim as if the interest rate were purely imaginary and the terminal condition resembles that of a default-free discount bond.

For a general Markov process $x(t)$, a closed-form solution for the above PDE subject to the terminal condition may not be available and numerical methods have to be used to solve the PDE. Nevertheless, the difficult problem of finding the density function of the sum has been transformed into a problem of solving a two-dimensional (including time dimension) PDE subject to a simple terminal condition.

For some of the most widely used stochastic processes in finance, like the Cox et al. (1985) square-root process for interest rates and the log-normal process for equity prices, tremendous simplifications are possible. For the former, because a closed-form solution for the discount bond price is known, a simple scaling of the parameters gives the solution of the above PDE. For the latter, because of the linearity of the stochastic differential equation $\mathrm{d}x = \mu x \mathrm{d}t + \sigma x \mathrm{d}w$, where μ and σ are constants, $F(x(0), \eta, t)$ can be obtained for different η's and t's by solving the PDE once.[8] Once the density function of the sum is obtained, Asian options can be priced using numerical integration, which is a very quick process.

2 Pricing of average Cox et al. (1985) square-root interest rate options

When the underlying process follows the Cox et al. (1985) square-root process, the density of the sum can be obtained analytically. Therefore closed-form solutions are available for average interest rate options.

Let the interest rate process under the risk-neutral measure Q be given by

$$\mathrm{d}r = \kappa(\theta - r)\mathrm{d}t + \sigma\sqrt{r}\mathrm{d}w^Q, \tag{12}$$

and the arithmetic sum of $r(u)$ from $u = s$ to $u = t$ be given by

$$y(t) = \int_s^t r(u)\mathrm{d}u. \tag{13}$$

For this interest rate process, the discount bond price at time s with maturity date t is given by

$$\begin{aligned}\Lambda(\kappa, \theta, \sigma, r(s), t - s) &= E_s^Q\left[e^{-\int_s^t r(u)\mathrm{d}u}\right] = \int_0^\infty e^{-y} f(y)\mathrm{d}y \\ &= A(t - s)e^{-B(t-s)r(s)},\end{aligned} \tag{14}$$

[7] The notation is to emphasize the fact that F is generally complex.

[8] For more detail, see Sect. 4.

where $f(\cdot)$ is the density function of $y(t)$, and $A(t-s)$ and $B(t-s)$ are given by

$$A(t-s) = \left(\frac{2\nu e^{(\kappa+\nu)(t-s)/2}}{(\kappa+\nu)(e^{\nu(t-s)}-1)+2\nu} \right)^{\frac{2\kappa\theta}{\sigma^2}},$$
$$B(t-s) = \frac{2(e^{\nu(t-s)}-1)}{(\kappa+\nu)(e^{\nu(t-s)}-1)+2\nu},$$

where $\nu = \sqrt{\kappa^2+2\sigma^2}$. Note that $\Lambda(\kappa,\theta,\sigma,r(s),t-s)$ satisfies the following fundamental evaluation PDE

$$-r\Lambda + \Lambda_s + \kappa(\theta - r)\Lambda_r + \frac{1}{2}\sigma^2 r \Lambda_{rr} = 0, \tag{15}$$

and the terminal condition

$$\Lambda(\kappa,\theta,\sigma,r(t),0) = 1.$$

Now consider

$$\hat{\Lambda}(\lambda, t-s) = E_s^Q\left[e^{-\lambda\int_s^t r(u)\mathrm{d}u}\right] = \int_0^\infty e^{-\lambda y} f(y)\mathrm{d}y \text{ for some real } \lambda \geq 0.$$

If one defines $z = \lambda r$, then

$$\hat{\Lambda}(\lambda, t-s) = E_s^Q\left[e^{-\int_s^t z(u)\mathrm{d}u}\right],$$

and $z(u)$ satisfies

$$\mathrm{d}z = \kappa(\lambda\theta - z)\mathrm{d}t + \sqrt{\lambda}\sigma\sqrt{z}\mathrm{d}w^Q.$$

This is the Cox et al. (1985) square-root process similar to $r(u)$ with different parameters, and $z(s) = \lambda r(s)$, so

$$\hat{\Lambda}(\lambda, t-s) = \int_0^\infty e^{-\lambda y} f(y)\mathrm{d}y = \Lambda(\kappa, \lambda\theta, \sqrt{\lambda}\sigma, \lambda r(s), t-s). \tag{16}$$

Finally, consider

$$F(r(s),\eta,t-s) = E_s^Q\left[e^{-i\eta\int_s^t r(u)\mathrm{d}u}\right] = \int_0^\infty e^{-i\eta y} f(y)\mathrm{d}y.$$

Since both $\int_0^\infty e^{-\lambda y} f(y)\mathrm{d}y$ and $\int_0^\infty e^{-i\eta y} f(y)\mathrm{d}y$ are integrable (In fact, $|\int_0^\infty e^{-i\eta y} f(y)\mathrm{d}y| \leq 1$.), they must have the same functional forms in terms of λ and $i\eta$.

Therefore

$$F(r(s),\eta,t-s)=E_s^Q\left[e^{-i\eta\int_s^t r(u)\mathrm{d}u}\right]=\int_0^\infty e^{-i\eta y}f(y)\mathrm{d}y$$
$$=\Lambda(\kappa,i\eta\theta,\sqrt{i\eta}\sigma,i\eta r(s),t-s)$$
$$=\tilde{A}(t-s)e^{-i\eta\tilde{B}(t-s)r(s)}, \tag{17}$$

where

$$\tilde{A}(t-s)=\left(\tilde{Z}(t-s)\right)^{\frac{2\kappa\theta}{\sigma^2}}, \tag{18}$$

$$\tilde{Z}(t-s)=\frac{2\nu e^{(\kappa+\nu)(t-s)/2}}{(\kappa+\nu)(e^{\nu(t-s)}-1)+2\nu}, \tag{19}$$

$$\tilde{B}(t-s)=\frac{2(e^{\nu(t-s)}-1)}{(\kappa+\nu)(e^{\nu(t-s)}-1)+2\nu}, \tag{20}$$

and $\nu=\sqrt{\kappa^2+2i\eta\sigma^2}$. It should be pointed out that even though ν is double-valued, both $\tilde{Z}(t-s)$ and $\tilde{B}(t-s)$ are single-valued functions. When $\frac{2\kappa\theta}{\sigma^2}$ is not an integer, $\tilde{A}(t-s)$ is multi-valued and its principal branch should be chosen. These properties are verified in Appendix A.

It is easy to check that $F(r(s),\eta,t-s)$ satisfies the PDE

$$-i\eta rF+F_s+\kappa(\theta-r)F_r+\frac{1}{2}\sigma^2 rF_{rr}=0,$$

which is derived in the previous section, and the terminal condition

$$F(r(t),\eta,0)=(1,0).$$

It is worthwhile to emphasize again the similarities between the Fourier transformation of the arithmetic sum $F(r(s),\eta,t-s)$ and the discount bond price $P(\kappa,\theta,\sigma,r(s),t-s)$. The analytical formulas for them are similar. They also satisfy similar PDE's and terminal conditions. Recognizing the similarity between them is the key insight of this paper which allows one to price Asian options in an efficient and novel way.

It follows from the previous section that the density function is given by

$$f(y)=\frac{2}{\pi}\int_0^\infty\cos(\eta y)R(F(r(s),\eta,t-s))\mathrm{d}\eta. \tag{21}$$

This density function of the sum of the interest rate process can be used to price Asian interest rate derivatives. Assume the current time is t, and the averaging period is from T' to T, so the payoff of a European Asian put at T is given by $p(T)=(K-A(T))^+$, where

$$A(T)=\int_{T'}^T r(u)\mathrm{d}u.$$

Two cases need to be considered separately: (1) time-to-maturity is shorter than the length of the averaging period, and (2) time-to-maturity is longer than the length of the averaging period. Only Asian puts will be considered. The prices of otherwise identical calls can be obtained from the put-call parities.

2.1 Case I: time-to-maturity is shorter than the length of the averaging period

When $t \geq T'$,

$$A(T) = \int_{T'}^{t} r(u)\mathrm{d}u + \int_{t}^{T} r(u)\mathrm{d}u = A(t) + \int_{t}^{T} r(u)\mathrm{d}u.$$

Using the risk-neutral pricing methods of Cox and Ross (1975), and Harrison and Kreps (1979), the current price of the put is given by

$$\begin{aligned} p(t) &= E_t^Q\left[e^{-\int_t^T r(u)\mathrm{d}u}(K - A(t) - \int_t^T r(u)\mathrm{d}u)^+\right] \\ &= \int_0^\infty e^{-y}\,(K - A(t) - y)^+\, f(y)\mathrm{d}y, \end{aligned} \tag{22}$$

where $f(y)$ is the density function of $y = \int_t^T r(u)\mathrm{d}u$.

2.1.1 Simplifications

When $A(t) \geq K$, the Asian put is going to finish out of the money for sure and therefore

$$p(t) = 0 \quad \text{if} \quad A(t) \geq K. \tag{23}$$

When $A(t) < K$, the pricing of an Asian put can be reduced to a one-dimensional integral. First note that

$$f(y) = \frac{2}{\pi}\int_0^\infty \cos(\eta y) R(F(r(t), \eta, T - t))\mathrm{d}\eta.$$

Therefore

$$\begin{aligned} p(t) &= \frac{2}{\pi}\int_0^\infty \int_0^\infty e^{-y}\,(K - A(t) - y)^+ \cos(\eta y) R(F(r(t), \eta, T - t))\mathrm{d}\eta\mathrm{d}y \\ &= \frac{2}{\pi}\int_0^\infty R(F(r(t), \eta, T - t))\mathrm{d}\eta \int_0^{K - A(t)} e^{-y}\,(K - A(t) - y)\cos(\eta y)\mathrm{d}y \\ &= \frac{2}{\pi}\int_0^\infty R(F(r(t), \eta, T - t)) I(\eta)\mathrm{d}\eta, \end{aligned} \tag{24}$$

where

$$I(\eta) = \int_0^{K-A(t)} e^{-y}(K - A(t) - y)\cos(\eta y)\mathrm{d}y = \frac{e^{-(K-A(t))}}{(1+\eta^2)^2}$$
$$\times\left[(1-\eta^2)\cos(\eta(K - A(t))) - 2\eta\sin(\eta(K - A(t)))\right]$$
$$+\frac{1}{1+\eta^2}\left(K - A(t) - \frac{1-\eta^2}{1+\eta^2}\right).$$

2.1.2 Put-call parity

The terminal payoffs of the call and put are related by

$$c(T) = (A(T) - K)^+ = (A(T) - K) + (K - A(T))^+ = A(T) - K + p(T).$$

Therefore,

$$\begin{aligned} c(t) &= E_t^Q\left[e^{-\int_t^T r(u)\mathrm{d}u}c(T)\right] = E_t^Q\left[e^{-\int_t^T r(u)\mathrm{d}u}(A(T) - K)\right] + p(t) \\ &= E_t^Q\left[e^{-\int_t^T r(u)\mathrm{d}u}\left(A(t) + \int_t^T r(u)\mathrm{d}u - K\right)\right] + p(t) \\ &= \Lambda(r(t), T - t)(A(t) - K) + V(r(t), T - t) + p(t), \end{aligned} \tag{25}$$

where $\Lambda(r(t), T - t)$[9] is the discount bond price at time t with maturity date T and $V(r(t), T - t)$ is given by

$$V(r(t), T_t) = E_t^Q\left[e^{-\int_t^T r(u)\mathrm{d}u}\int_t^T r(u)\mathrm{d}u\right].$$

$V(r(t), T - t)$ can be evaluated in closed-form using the following manipulation:

$$\begin{aligned} V =& -\frac{\partial}{\partial\lambda}E_t^Q\left[e^{-\lambda\int_t^T r(u)\mathrm{d}u}\right]|_{\lambda=1} = -\frac{\partial}{\partial\lambda}\hat{\Lambda}(\lambda, T-t)|_{\lambda=1} = \Lambda(r(t), T-t)\Bigg[B(T-t)r(t) \\ &\times\left(1+\frac{\sigma^2(T-t)e^{\nu(T-t)}}{\nu(e^{\nu(T-t)}-1)} - \frac{\sigma^2(e^{\nu(T-t)}-1) + (\kappa+\nu)\sigma^2(T-t)e^{\nu(T-t)} + 2\sigma^2}{\nu((\kappa+\nu)(e^{\nu(T-t)}-1) + 2\nu)}\right) \\ &-\frac{2\kappa\theta}{\nu}\left(\frac{1}{\nu} + \frac{T-t}{2} - \frac{(e^{\nu(T-t)}-1) + (\kappa+\nu)(T-t)e^{\nu(T-t)} + 2}{(\kappa+\nu)(e^{\nu(T-t)}-1) + 2\nu}\right)\Bigg]. \end{aligned} \tag{26}$$

[9] For simplicity, other parameters are omitted in the bond price formula except time-to-maturity and spot interest rate.

2.2 Case II: time-to-maturity is longer than the length of the averaging period

When $t < T'$, the payoff of an Asian put at the terminal date is

$$p(T) = \left(K - \int_{T'}^{T} r(u)\mathrm{d}u\right)^{+}.$$

Thus the price of the put at time t is

$$\begin{aligned} p(t) &= E_t^Q\left[e^{-\int_t^T r(u)\mathrm{d}u}\left(K - \int_{T'}^{T} r(u)\mathrm{d}u\right)^{+}\right] \\ &= E_t^Q\left[e^{-\int_t^{T'} r(u)\mathrm{d}u} E_{T'}^Q\left[e^{-\int_{T'}^{T} r(u)\mathrm{d}u}\left(K - \int_{T'}^{T} r(u)\mathrm{d}u\right)^{+}\right]\right], \end{aligned} \tag{27}$$

where the second equality follows from the law of iterative expectations. Let

$$H(r(T')) = E_{T'}^Q\left[e^{-\int_{T'}^{T} r(u)\mathrm{d}u}\left(K - \int_{T'}^{T} r(u)\mathrm{d}u\right)^{+}\right] = \int_0^\infty e^{-y}(K-y)^+ f(y)\mathrm{d}y, \tag{28}$$

where $f(y)$ is the density function of $y = \int_{T'}^{T} r(u)\mathrm{d}u$.

Using the forward risk-neutral measure R which is derived in Appendix B, Eq. 27 can be rewritten as

$$\begin{aligned} p(t) &= E_t^Q\left[e^{-\int_t^{T'} r(u)\mathrm{d}u} H(r(T'))\right] = \Lambda(r(t), T'-t) E_t^R\left[H(r(T'))\right] \\ &= \Lambda(r(t), T'-t)\int_0^\infty H(z)g(z)\mathrm{d}z, \end{aligned} \tag{29}$$

where $g(z)$, given in Appendix C, is the density function of $r(T')$ under the forward risk-neutral measure R.

2.2.1 Simplifications

Since

$$f(y) = \frac{2}{\pi}\int_0^\infty \cos(\eta y) R(F(r(t), \eta, T-t))\mathrm{d}\eta,$$

it follows that

$$\begin{aligned} H(r(T')) &= \frac{2}{\pi}\int_0^\infty R(F(r(t), \eta, T-t))\mathrm{d}\eta \int_0^K e^{-y}(K-y)\cos(\eta y)\mathrm{d}y \\ &= \frac{2}{\pi}\int_0^\infty R(F(r(t), \eta, T-t)) I(\eta)\mathrm{d}\eta, \end{aligned} \tag{30}$$

where

$$I(\eta) = \int_0^K e^{-y}(K-y)\cos(\eta y)\mathrm{d}y = \frac{e^{-K}}{(1+\eta^2)^2}\Big[(1-\eta^2)\cos(\eta K) \\ -2\eta\sin(\eta K)\Big] + \frac{1}{1+\eta^2}\left(K - \frac{1-\eta^2}{1+\eta^2}\right).$$

2.2.2 Put-call parity

$$\begin{aligned} c(t) &= E_t^Q\Big[e^{-\int_t^T r(u)\mathrm{d}u}c(T)\Big] = E_t^Q\Big[e^{-\int_t^T r(u)\mathrm{d}u}(A(T)-K)\Big] + p(t) \\ &= p(t) - \Lambda(r(t), T-t)K + E_t^Q\left[e^{-\int_t^T r(u)\mathrm{d}u}\int_{T'}^T r(u)\mathrm{d}u\right] \\ &= p(t) - \Lambda(r(t), T-t)K + E_t^Q\left[e^{-\int_t^{T'} r(u)\mathrm{d}u}E_{T'}^Q\left[e^{-\int_{T'}^T r(u)\mathrm{d}u}\int_{T'}^T r(u)\mathrm{d}u\right]\right] \\ &= p(t) - \Lambda(r(t), T-t)K + E_t^Q\Big[e^{-\int_t^{T'} r(u)\mathrm{d}u}V(r(T'), T-T')\Big] \\ &= p(t) - \Lambda(r(t), T-t)K + \Lambda(r(t), T'-t)\int_0^\infty V(z, T-T')g(z)\mathrm{d}z, \end{aligned} \quad (31)$$

where $V(r(T'), T-T')$ is given in Eq. 26 and $g(z)$ in Appendix C.

2.2.3 Computation efficiency consideration

For interest rates and foreign exchange rates, there are reasons to believe that the Cox et al. (1985) square-root process is a more appropriate description than the log-normal process. First, the expectation of the future value of a log-normal process increases exponentially. But it is very unlikely that the expected future interest rates or the expected future foreign exchange rates will increase monotonically. The Cox et al. (1985) square-root process also has the desired property that it is mean-reverting. Nevertheless, most of the work on the pricing of Asian options, whether they are equity Asian options, or interest rate Asian options, or currency Asian options, etc, has assumed the underlying process to be log-normal for tractability. In cases where the Cox et al. (1985) square-root process is more appropriate, except the Monte Carlo simulation of Kemna and Vorst (1990), none of the methods mentioned in the introduction is applicable.[10]

In the case of Sect. 3.1, the usual PDE approach would have to solve a three-dimensional (including the time dimension) PDE once to obtain the option price in Eq. 24. In the present approach, the computation is reduced to a one-dimensional integral (see Eq. 24), a one-dimensional problem. The dimensionality of the problem has been reduced by two.

[10] Besides Monte Carlo simulation, the usual PDE approach is also applicable. But the variable reduction technique of Rogers and Shi (1995) would not apply.

In the case of Sect. 3.2, the usual PDE approach would have to solve a three-dimensional PDE many times to obtain $H(r(T'))$ (see Eq. 28) for different $r(T')$. These $H(r(T'))$'s are needed to obtain the option price in Eq. 29. In the usual PDE approach, the expectation in Eq. 29 can be obtained by solving a two-dimensional PDE using the $H(r(T'))$'s as the terminal values. Therefore, the PDE approach in this case is essentially a four-dimensional problem (solving a three-dimensional PDE many times). On the other hand, the present approach involves only a two-dimensional integral (see Eqs. 29 and 30), a two-dimensional problem. Therefore in this case too the present approach has reduced the dimensionality of the problem by two.

3 Pricing of average log-normal equity options

The pricing of equity Asian options is considered in this section. Throughout this section the Black and Scholes (1973) economy in which the riskless interest rate is a constant and the underlying equity price follows a log-normal process is assumed. Let the process under the risk-neutral measure Q be given by[11]

$$\mathrm{d}S(t) = rS(t)\mathrm{d}t + \sigma S(t)\mathrm{d}w^Q(t), \tag{32}$$

where $w^Q(t)$ is standard Wiener process under Q.

Assume the current time is t, and the averaging period is from T' to T, so the payoff of a European Asian put at T is given by $P(T) = (K - A(T))^+$, where

$$A(T) = \int_{T'}^{T} S(u)\mathrm{d}u.$$

Again, two cases need to be considered: (1) time-to-maturity is shorter than the length of the averaging period, and (2) time-to-maturity is longer than the length of the averaging period. Only Asian puts will be considered. The prices of otherwise identical calls will be given in terms of put-call parities.

3.1 Case I: time-to-maturity is shorter than the length of the averaging period

When $t \geq T'$,

$$A(T) = \int_{T'}^{t} S(u)\mathrm{d}u + \int_{t}^{T} S(u)\mathrm{d}u = A(t) + \int_{t}^{T} S(u)\mathrm{d}u.$$

It will become clear that it is convenient and efficient to define $X(u) = S(u)/S(t)$. The price of an Asian put at time t is

$$\begin{aligned} P(t) &= e^{-r(T-t)} E_t^Q\left[(K - A(T))^+\right] \\ &= e^{-r(T-t)} \int_0^{\infty} (K - A(t) - S(t)y)^+ f(y)\mathrm{d}y. \end{aligned} \tag{33}$$

[11] It is assumed that the stock pays no dividend. A constant dividend yield can be easily incorporated.

where $f(\cdot)$ is the density function of $y = \int_t^T X(u)\mathrm{d}u$.

No analytical formula is known for the density function of y, numerical method has to be employed. In Sect. 4.3 a very efficient algorithm is presented for computing this density function.

3.1.1 Simplifications

If $A(t) \geq K$, the price of an Asian put is easily seen to be zero:

$$P(t) = 0 \quad \text{if} \quad A(t) \geq K. \tag{34}$$

If $A(t) < K$, Eq. 33 can be reduced to a one-dimensional integral. Let $F(\eta, T-t)$[12] be the Fourier transformation of $f(y)$, then

$$f(y) = \frac{2}{\pi}\int_0^{\infty} \cos(\eta y) R(F(\eta, T-t))\mathrm{d}\eta.$$

It follows that

$$\begin{aligned} P(t) &= \frac{2}{\pi}e^{-r(T-t)}\int_0^{\infty}\int_0^{\infty}(K - A(t) - S(t)y)^+ \cos(\eta y)R(F(\eta, T-t))\mathrm{d}\eta\mathrm{d}y \\ &= \frac{2}{\pi}e^{-r(T-t)}\int_0^{\infty}R(F(\eta, T-t))\mathrm{d}\eta\int_0^{\frac{K-A(t)}{S(t)}}(K - A(t) - S(t)y)\cos(\eta y)\mathrm{d}y \\ &= \frac{2}{\pi}e^{-r(T-t)}\int_0^{\infty}R(F(\eta, T-t))I(\eta)\mathrm{d}\eta, \end{aligned} \tag{35}$$

where

$$I(\eta) = \int_0^{\frac{K-A(t)}{S(t)}}(K - A(t) - S(t)y)\cos(\eta y)\mathrm{d}y = \frac{S(t)}{\eta^2}\left(1 - \cos\frac{\eta(K - A(t))}{S(t)}\right).$$

3.1.2 Put-call parity

The terminal payoff functions of the call and put are related by

$$C(T) = (A(T) - K)^+ = (A(T) - K) + (K - A(T))^+ = (A(T) - K) + P(T).$$

Discounting, one has

$$\begin{aligned} C(t) &= e^{-r(T-t)}E_t^Q\left[(A(T) - K)^+\right] = e^{-r(T-t)}E_t^Q[A(T) - K] + P(t) \\ &= e^{-r(T-t)}E_t^Q[A(T)] - e^{-r(T-t)}K + P(t) \end{aligned}$$

[12] Since the $X(u) = S(u)/S(t)$ $(u \geq t)$ process always starts at $X(t) = 1$, the initial value of $X(t)$ is omitted in $F(\eta, T-t)$.

Since $A(T) = A(t) + \int_t^T S(u)\mathrm{d}u$,

$$E_t^Q[A(T)] = A(t) + \int_t^T E_t^Q[S(u)]\mathrm{d}u = A(t) + S(t)(e^{r(T-t)} - 1)/r.$$

Therefore, the prices of an Asian call and an otherwise identical put are related by

$$C(t) = P(t) + e^{r(T-t)}(A(t) - K) + S(t)(1 - e^{-r(T-t)})/r. \tag{36}$$

3.1.3 Hedging parameters

Because of the explicit form of $P(t)$ in terms of $S(t)$ and $A(t)$, the portfolio hedging parameters can be easily obtained:

$$\begin{aligned}\Delta = \partial P/\partial S = \frac{2}{\pi}e^{-r(T-t)}\int_0^\infty R(F(\eta, T-t))\left[\frac{1}{\eta^2}(1 - \cos\left(\frac{\eta(K - A(t))}{S(t)}\right)\right.\\ \left. - \frac{K - A(t)}{\eta S(t)}\sin\left(\frac{\eta(K - A(t))}{S(t)}\right)\right]\mathrm{d}\eta,\\ \Gamma = \partial^2 P/\partial^2 S = \frac{2}{\pi}e^{-r(T-t)}\int_0^\infty R(F(\eta, T-t))\frac{(K - A(t))^2}{S(t)^3}\cos\frac{\eta(K - A(t))}{S(t)}\mathrm{d}\eta.\end{aligned}$$

To compute $\Theta = \partial P/\partial t$, one can make use of the PDE

$$-rP + P_t + rSP_S + \frac{1}{2}\sigma^2 S^2 P_{SS} + SP_A = 0, \tag{37}$$

satisfied by P because P_A can be easily calculated:

$$\partial P/\partial A = -\frac{2}{\pi}e^{-r(T-t)}\int_0^\infty R(F(\eta, T-t))\frac{1}{\eta}\sin\frac{\eta(K - A(t))}{S(t)}\mathrm{d}\eta.$$

To obtain other portfolio parameters such as $\rho = \partial P/\partial r$ and $\Lambda = \partial P/\partial \sigma$, the density function $f(y)$ (actually $F(\eta, T-t)$) has to be calculated at least twice using slightly different input parameters such as r and σ. Fortunately, these portfolio parameters are not used as widely as Δ, Γ and Θ.

3.2 Case II: time-to-maturity is longer than the length of the averaging period

When $t < T'$,

$$\begin{aligned} P(t) &= e^{-r(T-t)} E_t^Q\big[(K - A(T))^+\big] = e^{-r(T-t)} E_t^Q\left[(K - \int_{T'}^{T} S(u)\mathrm{d}u)^+\right] \\ &= e^{-r(T-t)} E_t^Q\left[E_{T'}^Q\left[(K - S(T')\int_{T'}^{T} X(u)\mathrm{d}u)^+\right]\right] \\ &= e^{-r(T-t)} E_t^Q\left[\int_0^\infty (K - S(T')y)^+ f(y)\mathrm{d}y\right], \end{aligned} \tag{38}$$

where $X(u) = S(u)/S(T')$ and $f(\cdot)$ is the density function of $y = \int_{T'}^{T} X(u)\mathrm{d}u$.

3.2.1 Simplifications

Since

$$S(T') = S(t)e^{(r-\sigma^2/2)(T'-t)+\sigma(w^Q(T')-w^Q(t))} = S(t)e^u,$$

where $u = (r - \sigma^2/2)(T' - t) + \sigma(w^Q(T') - w^Q(t))$ is normally distributed with mean $(r - \sigma^2/2)(T' - t)$ and variance $\sigma^2(T' - t)$, $P(t)$ can be rewritten as

$$\begin{aligned} P(t) &= e^{-r(T-t)} \int_{-\infty}^{\infty} h(u)\left(\int_0^{\frac{K}{S(t)e^u}} (K - S(t)e^u y) f(y)\mathrm{d}y\right)\mathrm{d}u \\ &= \frac{2}{\pi} e^{-r(T-t)} \int_{-\infty}^{\infty} h(u)\mathrm{d}u \int_0^\infty R(F(\eta, T - T'))\mathrm{d}\eta \\ &\quad \times \int_0^{\frac{K}{S(t)e^u}} \cos(\eta y)(K - S(t)e^u y)\mathrm{d}y \\ &= \frac{2}{\pi} e^{-r(T-t)} \int_{-\infty}^{\infty}\int_0^\infty h(u) R(F(\eta, T - T')) I(u, \eta)\mathrm{d}u\mathrm{d}\eta, \end{aligned} \tag{39}$$

where $h(\cdot)$ is the density function of u, $R(F(\eta, T - T'))$ is the real part of the Fourier transformation $F(\eta, T - T')$ of $f(y)$ and

$$I(u, \eta) = \int_0^{\frac{K}{S(t)e^u}} \cos(\eta y)(K - S(t)e^u y)\mathrm{d}y = \frac{S(t)e^u}{\eta^2}\left(1 - \cos\frac{\eta K}{S(t)e^u}\right).$$

Therefore Eq. 38 has been reduced to a two-dimensional integral.

3.2.2 Put-call parity

Again,

$$C(t) = e^{-r(T-t)} E_t^Q[A(T)] - e^{-r(T-t)} K + P(t).$$

Since $A(T) = \int_{T'}^{T} S(u)\mathrm{d}u$,

$$E_t^Q[A(T)] = \int_{T'}^{T} E_t^Q[S(u)]\mathrm{d}u = S(t)(e^{r(T-t)} - e^{r(T'-t)})/r.$$

Therefore, the prices of an Asian call and an otherwise identical put are related by

$$C(t) = P(t) - e^{r(T-t)} K + S(t)(1 - e^{-r(T-T')})/r. \tag{40}$$

3.2.3 Hedging parameters

Because of the explicit form of $P(t)$ as a function of $S(t)$, the portfolio parameters Δ, Γ and Θ can be easily calculated.

$$\begin{aligned}\Delta = \partial P/\partial S &= \frac{2}{\pi} e^{-r(T-t)} \int_{-\infty}^{\infty}\int_{0}^{\infty} h(u)R(F(\eta, T-T'))\\ &\times \left[\frac{e^u}{\eta^2}\left(1 - \cos\frac{\eta K}{S(t)e^u}\right) - \frac{K}{\eta S(t)} \sin\frac{\eta K}{S(t)e^u}\right]\mathrm{d}u\mathrm{d}\eta,\\ \Gamma = \partial^2 P/\partial^2 S &= \frac{2}{\pi} e^{-r(T-t)} \int_{-\infty}^{\infty}\int_{0}^{\infty} h(u)R(F(\eta, T-T'))\\ &\times \frac{K^2}{S(t)^3 e^u}\cos\left(\frac{\eta K}{S(t)e^u}\right)\mathrm{d}u\mathrm{d}\eta.\end{aligned}$$

To compute $\Theta = \partial P/\partial s$, one can make use of the PDE

$$-rP + P_s + rSP_S + \frac{1}{2}\sigma^2 S^2 P_{SS} = 0, \tag{41}$$

satisfied by P when $t \le s < T'$. Again, to compute $\rho = \partial P/\partial r$ and $\Lambda = \partial P/\partial \sigma$, $f(y)$ (i.e., $F(\eta, T - T')$) has to be calculated at least twice using slightly different inputs r and σ.

3.3 Computing the density function

It is now time to show how to obtain the density function of the sum of a log-normal process. As it has been shown, one only needs to consider the log-normal process $X(t) = S(t)/S(0)$. That is, one only needs to consider those log-normal processes which always start from $X(0) = 1$.

It has been shown in Sect. 2 that the characteristic function $F(X(t), \eta, T-t)$ of the sum of a Markov process satisfies a PDE. In the present case, it is

$$-i\eta XF + F_t + rXF_X + \frac{1}{2}\sigma^2 X^2 F_{XX} = 0. \tag{42}$$

The corresponding terminal condition is

$$F(X(T), \eta, 0) = (1, 0).$$

No closed-form solution is known for this PDE subject to the terminal condition. Numerical methods have to be employed to solve it.

Because of the linearity of the stochastic differential equation (SDE)

$$\mathrm{d}X(t) = rX(t)\mathrm{d}t + \sigma X(t)\mathrm{d}w^Q(t),$$

it should be noted that, in functional form, the solution to the above PDE has to be of $F(\eta X(t), T-t)$.[13] That is η and $X(t)$ always appear together as a product. This can be easily seen from a change of variable $Y(t) = \eta X(t)$. In terms of Y, the PDE and terminal condition are given by

$$-iYF + F_t + rYF_Y + \frac{1}{2}\sigma^2 Y^2 F_{YY} = 0,$$

and

$$F(Y(T), 0) = (1, 0).$$

Therefore η and $X(t)$ always appear together as one argument $Y = \eta X$. This property drastically speeds up the process of obtaining the Fourier transformation of the density function of the sum as it will be seen momentarily.

The implicit finite difference method is suggested to be used to solve the above PDE.[14] The usual change of variable $Z = \log(X)$ will not be made because an evenly spaced grid in X (i.e., evenly in η, see below.) facilitates the inversion of the Fourier transformation to get the density function. The implicit finite difference method is chosen because it ensures convergence.

Let $X = X_{\min} = 0$ and $X = X_{\max}$ represent the lower and upper boundaries in the grid for X. To solve the PDE, besides the terminal condition $F(\eta X(T), 0) = (1, 0)$, two boundary conditions at $X = X_{\min}$ and $X = X_{\max}$ need to be specified. It should be clear that

$$F(\eta X_{\min} = 0, T-t) = (1, 0) \quad \text{for any } 0 \le t \le T$$

[13] Without confusion, the same F is used to represent the new functional form.

[14] It is noted that solving the system of equations in an implicit finite difference scheme does not involve inverting a matrix. In fact, the system of equations can be solved in $O(N)$ steps, where N is the number of discrete values of X in the grid. For detail, see Courtadon (1982).

because, when $X(t) = 0$, the process is absorbed at the origin and $y = \int_t^T X(u)\mathrm{d}u = 0$. For the boundary condition at $X = X_{\max}$, one can make use of the Riemann-Lebesgue Lemma.[15] Therefore,

$$F(\eta X_{\max}, T - t) = (0, 0) \quad \text{for any } 0 \le t \le T.$$

For a given η, by working backwards to $t = 0$, one not only gets $F(\eta X_0, T) = F(\eta, T)$ corresponding to initial value $X(0) = 1$, but also gets $F(\eta X_i, T)$ corresponding to initial process value X_i in the grid. In most of the derivative pricing applications using a finite difference method, the values in the grid at the initial date corresponding to the underlying process values other than its initial value are wasted. This seemingly waste of calculations turns out to save a lot of computations in the present application.

If one reinterprets $F(\eta X_i, T)$ as the value of a new Fourier component corresponding to initial process value $X_0 = X(0) = 1$ and a new Fourier component ηX_i, then one will have as many different Fourier components as the number of discrete values of X in the grid. Thus by solving the PDE (grid) once, one will be able to get the whole spectrum of the Fourier transformation, thanks to the linearity of the process

$$\mathrm{d}X = rX\mathrm{d}t + \sigma X\mathrm{d}w.$$

It is clear now why an evenly spaced grid in X is chosen. After the scaling, one will have the Fourier components ηX_i evenly spaced with step size $\eta\Delta X$ for a given η, where ΔX is the step size of X in the grid. Fourier inverse transformation of $F(\eta_i, T)$, where $\eta_i = \eta X_i$, yields the density function of the sum with initial process value $X(0) = 1$. Of course, one could have begun with $\eta = 1$. Then the value of $F(X_i, T)$ in the grid corresponding to the initial date $t = 0$ will be the Fourier transformation corresponding to the Fourier component $\eta_i = X_i$.

3.4 Tabulation of the density function

In fact, the Fourier transformations for shorter averaging periods are also obtained. This is because the values at every time step have been calculated, and for whatever averaging period, the initial value $X(0)$ is always 1. Therefore, if one solves the corresponding PDE for the sum of averaging length T, one will also obtain the Fourier transformations for sums of all averaging lengths $< T$, all with the initial value $X(0) = 1$. It is worthwhile to emphasize that the density function $f(\cdot)$ of the sum of $X(\cdot)$ will be a function of the length of the sum interval $T - t$ only (besides r and σ).[16] The main result is that by solving the PDE once, one will be able to get the density function of the sum of $X(\cdot)$ for different summation periods. For a given stock, the density function of the sum of $X(u) = S(u)/S(0)$ needs only to be calculated once and can be stored for later use. For example, at any date t, the density function of the

[15] Riemann–Lebesgue Lemma states that if $f(x)$ is Riemann integrable, then $\lim_{\alpha\to\infty} \int_a^b f(x)\sin(\alpha x)\mathrm{d}x = \lim_{\alpha\to\infty} \int_a^b f(x)\cos(\alpha x)\mathrm{d}x = 0$. See Marsden and Hoffman (1993).

[16] It depends on the initial value $X(0)$, but $X(0) = 1$ always.

sum of $X(u) = S(u+t)/S(t)$ from $u = 0$ to $u = T - t$, no matter what stock price S_t is, is always given by $f(y)$, where $f(y)$ is the density function of $y = \int_0^{T-t} X(u)\mathrm{d}u$. Therefore, for a given stock, the density function of the sum can be tabulated according to different length of the averaging period.

3.5 Computation efficiency consideration

It is clear from the previous subsection (Sect. 4.3) that a two-dimensional (including the time dimension) PDE needs to be solved to obtain the Fourier transformation $F(\eta, T-t)$ of the density function of the sum of $X(\cdot)$, which always start with $X(t) = 1$. This is a two-dimensional problem.

In the case of Sect. 4.1, the present approach needs to compute the one-dimensional integral in Eq. 35. Therefore in this case, it is overall a two-dimensional method. On the other hand, the usual PDE approach solves a three-dimensional PDE (a three-dimensional method). The variable reduction technique of Rogers and Shi (1995) reduces the dimensionality of the PDE approach from three to two. Thus their method and the present approach should have comparable efficiency.

In the case of Sect. 4.2 (time-to-maturity is longer than the length of the averaging period), the present approach needs to compute the two-dimensional integral in Eq. 39. But once again, our approach is overall a two-dimensional method. This compares favorably to the three-dimensional method of the usual PDE approach and should have comparable efficiency as the two-dimensional approach of Rogers and Shi (1995).

4 Pricing of average-rate options in more general settings

It will be shown in this section that the method presented can be easily extended to the multiple state variable cases. Without loss of generality, for average interest rate derivatives, a two-factor model will be considered. For average equity options, a stochastic interest rate model will be considered.

First consider the two-factor average interest rate derivatives. Let

$$\begin{aligned} \mathrm{d}r_1 &= \mu_1(r_1, r_2, t)\mathrm{d}t + \eta_1(r_1, r_2, t)\mathrm{d}w_1^Q, \\ \mathrm{d}r_2 &= \mu_2(r_1, r_2, t)\mathrm{d}t + \eta_2(r_1, r_2, t)\mathrm{d}w_2^Q, \end{aligned} \tag{43}$$

where w_1^Q and w_2^Q are two standard Wiener processes with correlation coefficient ρ.

Suppose one wants to find the density function of $y(t) = \int_0^t r(u)\mathrm{d}u$, where $r = r_1 + r_2$. Consider the Fourier transformation of the density function of $y(t)$ (characteristic function of $y(t)$),

$$F(r_1(0), r_2(0), \eta, t) = E_0^Q\left[\exp\left(-i\eta\int_0^t r(u)\mathrm{d}u\right)\right] = \int_0^\infty e^{-i\eta y} f(y)\mathrm{d}y,$$

where $f(\cdot)$ is the density function of $y(t)$.

To find $F(r_1(0), r_2(0), \eta, t)$, a PDE similar to the one derived in Sect. 2 will be derived. To this end, define

$$F(r_1(s), r_2(s), \eta, t-s) = E_s^Q\left[\exp\left(-i\eta\int_s^t (r_1(u)+r_2(u))\mathrm{d}u\right)\right].$$

Because $F^*(r_1(s), r_2(s), k, t-s) = \exp(-i\eta\int_0^s (r_1(u)+r_2(u))\mathrm{d}u)F(r_1(s), r_2(s), \eta, t-s)$ is a martingale, Ito's lemma implies

$$-i\eta(r_1+r_2)F + F_s + \mu_1 F_{r_1} + \mu_2 F_{r_2} + \frac{1}{2}\eta_1^2 F_{r_1r_1} + \frac{1}{2}\eta_2^2 F_{r_2r_2} + \rho\eta_1\eta_2 F_{r_1r_2} = 0. \tag{44}$$

The terminal condition is

$$F(r_1(t), r_2(t), \eta, 0) = (1, 0).$$

If the discount bond price in such a two-factor model is known, scaling of the parameters will give the solution of the above PDE. When closed solutions are not available, numerical method has to be used. But still the dimensionality of the problem has been reduced by one.[17]

Consider now an equity average option when the risk-free rate is stochastic and correlated with the stock price and the variance of the return is stochastic. Let

$$\begin{aligned} \mathrm{d}S &= rS\mathrm{d}t + S\sigma(S,t)\mathrm{d}w_1^Q, \\ \mathrm{d}r &= \mu(r,t)\mathrm{d}t + \eta(r,t)\mathrm{d}w_2^Q, \end{aligned} \tag{45}$$

where w_1^Q and w_2^Q are two standard Wiener processes with correlation coefficient ρ.

Suppose one wants to find the density function of $y(t) = \int_0^t S(u)\mathrm{d}u$. If one defines

$$F(r(s), S(s), \eta, t-s) = E_s^Q\left[\exp\left(-i\eta\int_s^t S(u)\mathrm{d}u\right)\right],$$

$F(r(0), S(0), \eta, t)$ will be the Fourier transformation of the density function of $y(t)$. By now it is clear that $F(r(s), S(s), k, t-s)$ satisfies the following PDE:

$$-i\eta SF + F_s + rSF_S + \mu F_r + \frac{1}{2}\sigma^2 S^2 F_{SS} + \frac{1}{2}\eta^2 F_{rr} + \rho S\sigma\eta F_{Sr} = 0. \tag{46}$$

The terminal condition is

$$F(r(t), S(t), \eta, 0) = (1, 0).$$

[17] The PDE satisfied by the price of a derivative involves three state variables: $r_1(t)$, $r_2(t)$ and the running sum $y(t)$ so far.

It should be pointed out that if $\sigma(S, t)$ is a constant, the scaling technique used for the log-normal process would still work because of the linearity of the stochastic differential equation (SDE) satisfied by the stock price. Once the Fourier transformation of the sum is obtained, Fourier inverse transformation will yield the density function. Pricing of Asian options can be proceeded as in Sects. 3 and 4.

5 Summary and discussions

A flexible and efficient approach for pricing Asian options for general underlying processes has been proposed. This approach overcomes most of the drawbacks of the other approaches. Analytical formulas for Asian Cox et al. (1985) interest rate options are obtained. Analytical formulas can also be derived for the more general affine class of term-structure models. A very efficient numerical method is proposed for pricing Asian equity options when the underlying process is log-normal. Extensions of the techniques to the cases of multiple state variables have also been discussed.

Even though the presentation has only concentrated on the Cox et al. (1985) square-root interest rate Asian options and the log-normal equity Asian options, the techniques obviously apply to other Asian options such as currency options when the underlying follows either the Cox et al. (1985) square-root process or the log-normal process.

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Appendix A

We prove in this appendix that $\tilde{Z}$ and $\tilde{B}$ given in Eqs. 19 and 20 are single-valued functions, and the principal branch of the multivalued function $\tilde{A} = \tilde{Z}^{(2\kappa\theta/\sigma^2)}$ given in (18) should be chosen.

For functions $\tilde{Z}$ and $\tilde{B}$, the only possibility of multiplicity comes from the fact that $\nu = \sqrt{\kappa^2 + 2i\eta\sigma^2}$ is a double-valued function. Let $\nu = \pm(x + iy)$. Simple algebra will show that same results would obtain for $\tilde{Z}$ and $\tilde{B}$ by using either $\nu = x + iy$ or $\nu = -(x + iy)$. Therefore $\tilde{Z}$ and $\tilde{B}$ may look like multivalued functions, but they are not. However, when $\frac{2\kappa\theta}{\sigma^2}$ is not an integer, $\tilde{A} = \tilde{Z}^{(2\kappa\theta/\sigma^2)}$ is a multivalued complex function. We now show that to be consistent with the definition of $F(r(s), \eta, t - s)$ in Eq. 2, the principal branch of this multivalued function should be chosen.

From the definition

$$F(r(s), \eta, t - s) = \int_0^\infty e^{-i\eta y} f(y)\mathrm{d}y,$$

it is obvious that $F(r(s), \eta, t-s)$ is single-valued and as $\eta \longrightarrow 0$, $F(r(s), \eta, t-s) \longrightarrow (1, 0)$. Let $\tilde{Z} = |\tilde{Z}|e^{i\alpha}$ where $\alpha \in (-\pi, \pi]$. The multivalued function $\tilde{A} = \tilde{Z}^{(2\kappa\theta/\sigma^2)}$ is given by

$$\tilde{A} = |\tilde{Z}|^{(2\kappa\theta/\sigma^2)} e^{i(2\kappa\theta/\sigma^2)\alpha + i(2\kappa\theta/\sigma^2)2n\pi},$$

where each integer n defines a branch for the multivalued function $\tilde{A} = \tilde{Z}^{(2\kappa\theta/\sigma^2)}$. It is easy to check that as $\eta \longrightarrow 0$, $|\tilde{Z}| \longrightarrow 1$, and $\alpha \longrightarrow 0$. Therefore as $\eta \longrightarrow 0$,

$$\tilde{A} = \tilde{Z}^{(2\kappa\theta/\sigma^2)} \longrightarrow e^{i(2\kappa\theta/\sigma^2)2n\pi}.$$

But since $F(r(s), \eta, t-s) \longrightarrow (1, 0)$, and $e^{-i\eta\tilde{B}} \longrightarrow (1, 0)$, it must be the case that $\tilde{A} \longrightarrow (1, 0)$ as $\eta \longrightarrow 0$. It follows then that the branch with $n = 0$ should be chosen.[18]

Appendix B

The Girsanov theorem[19] will be used to derive the forward risk-neutral measure in this Appendix.

Suppose one wants to compute the following expectation

$$H(r(0), t = 0) = E_0^Q[e^{-\int_0^T r(u)\mathrm{d}u} H(\{r(\cdot)\}, T)], \tag{47}$$

where $\{r(\cdot)\}$ indicates that $H(\cdot, T)$ could depend on the sample path of $r(\cdot)$ from 0 to T and Q denotes the risk-neutral probability measure. Let $P(r(0), T)$ be the discount bond price at $t = 0$ with maturity T. Define

$$\xi_T = \frac{e^{-\int_0^T r(u)\mathrm{d}u}}{\Lambda(r(0), T)}.$$

Then

$$H(r(0), t = 0) = \Lambda(r(0), T) E_0^Q[\xi_T H(\{r(\cdot)\}, T)]. \tag{48}$$

It is clear that ξ_T is strictly positive and $E_0^Q[\xi_T] = 1$. Therefore it can be used as a Radon–Nikodym derivative to define a probability measure R equivalent to the original measure Q such that

$$E_0^R[1_{\{A\}}] = E_0^Q[\xi_T 1_{\{A\}}]$$

for any event A. Under the new measure R,

$$H(r(0), t = 0) = \Lambda(r(0), T) E_0^R[H(\{r(\cdot)\}, T)]. \tag{49}$$

[18] Most computers, if not all, implement multivalued complex functions using the principal branch.

[19] For more details on Girsanov theorem and related materials, see Karatzas and Shreve (1991).

To find R, define the likelihood ratio

$$\xi_t = E_t^Q[\xi_T] = \frac{e^{-\int_0^t r(s)\mathrm{d}s}\Lambda(r(t), T-t)}{\Lambda(r(0), T)}.$$

It follows that

$$\log \xi_t = -\int_0^t r(s)\mathrm{d}s + \log \Lambda(r(t), T-t) - \log \Lambda(r(0), T).$$

Ito's lemma implies

$$\begin{aligned}
\mathrm{d}\log \xi_t &= -r\mathrm{d}t + \frac{\mathrm{d}\Lambda(r(t), T-t)}{\Lambda(r(t), T-t)} - \frac{1}{2}\left(\frac{\mathrm{d}\Lambda(r(t), T-t)}{\Lambda(r(t), T-t)}\right)^2 \\
&= \left(-r\Lambda + \Lambda_t + u(r)\Lambda_r + \frac{1}{2}\sigma^2(r)\Lambda_{rr}\right)\mathrm{d}t/\Lambda + \frac{\sigma(r)\Lambda_r}{\Lambda}\mathrm{d}w^Q \\
&\quad -\frac{1}{2}\left(\frac{\sigma(r)\Lambda_r}{\Lambda}\right)^2 \mathrm{d}t.
\end{aligned}$$

The term inside the bracket is the fundamental PDE satisfied by discount bond price Λ, thus

$$\mathrm{d}\log \xi_t = -\frac{1}{2}\left(\frac{\sigma(r)\Lambda_r}{\Lambda}\right)^2 \mathrm{d}t + \frac{\sigma(r)\Lambda_r}{\Lambda}\mathrm{d}w^Q.$$

Another application of Ito's lemma yields

$$\frac{\mathrm{d}\xi_t}{\xi_t} = \frac{\sigma(r)\Lambda_r}{\Lambda}\mathrm{d}w^Q.$$

Now Girsonov's theorem implies that

$$w^R(t) = w^Q(t) - \int_0^t \frac{\sigma(r,u)\Lambda_r(r, T-u)}{\Lambda(r, T-u)}\mathrm{d}u \tag{50}$$

is a standard Wiener process under R. In the differential form, one has

$$\mathrm{d}w^R = \mathrm{d}w^Q - \frac{\sigma(r)\Lambda_r}{\Lambda}\mathrm{d}t. \tag{51}$$

Appendix C

The result in Appendix B will be applied to the Cox et al. (1985) square-root interest process in this appendix. Suppose the interest rate process under the risk-neutral measure Q is given by

$$dr = \kappa(\theta - r)dt + \sigma\sqrt{r}dw^Q. \tag{52}$$

For this interest rate process, the discount bond price at time t with maturity date T is given by

$$\Lambda(r(t), T - t) = A(T - t)e^{-B(T-t)r(t)}, \tag{53}$$

where A and B are given in Sect. 3. From Eq. 5 in Appendix A it follows that under the forward risk-neutral probability measure R,

$$dw^Q = dw^R + \frac{\sigma(r)\Lambda_r}{\Lambda}dt = dw^R - \sigma\sqrt{r}B(T - t)dt. \tag{54}$$

Therefore under R the interest rate process is given by

$$\begin{aligned} dr &= \kappa(\theta - r)dt + \sigma\sqrt{r}(dw^R - \sigma\sqrt{r}B(T - t)dt) \\ &= (\kappa\theta - (\kappa + \sigma^2 B(T - t))r)dt + \sigma\sqrt{r}dw^R. \end{aligned} \tag{55}$$

The density function of $r(T)$ at time T conditional on $r(t)$ at time t is the noncentral χ^2 function:[20]

$$g(z) = (\phi + \psi)e^{-(\phi+\psi)z-\lambda}\left(\frac{(\phi + \psi)z}{\lambda}\right)^{\frac{q}{2}} I_q(2\sqrt{\lambda(\phi + \psi)z}), \tag{56}$$

where $I_q(x)$ is the modified Bessel function of the first kind, $\nu = \sqrt{\kappa^2 + 2\sigma^2}$ and

$$\begin{aligned} q &= \tfrac{2\kappa\theta}{\sigma^2} - 1, & \lambda &= \tfrac{\phi^2 e^{\nu(T-t)}r(t)}{\phi+\psi}, \\ \phi &= \tfrac{2\nu}{\sigma^2(e^{\nu(T-t)}-1)}, & \psi &= \tfrac{\kappa+\nu}{\sigma^2}. \end{aligned}$$

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