Pricing an American Option by Approximating Its Early Exercise Boundary as a Multipiece Exponential Function

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This article proposes to price an American option by approximating its early exercise boundary as a multipiece exponential function. Closed form formulas are obtained in terms of the bases and exponents of the multipiece exponential function. It is demonstrated that a three-point extrapolation scheme has the accuracy of an 800-time-step binomial tree, but is about 130 times faster. An intuitive argument is given to indicate why this seemingly crude approximation works so well. Our method is very simple and easy to implement. Comparisons with other leading competing methods are also included.

Pricing and hedging American options have been challenging problems. McKean (1965) and Merton (1973) demonstrate that the pricing of American options is a free boundary problem. The difficulty in pricing such options stems from the possibility of early exercise, and the early exercise boundary must be determined as part of the solution. Even though it is highly desirable, a closed form formula has not been found and it is not likely that one will be found anytime soon. Therefore efforts have been concentrated on approximate methods. Numerical methods such as the finite difference method of Brennan and Schwartz (1977) and the binomial tree model of Cox, Ross, and

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Rubinstein (1979) are among the earliest approximate methods and are still widely used. Even though these numerical methods are quite flexible and simple to implement, they are very time consuming. Academics and practitioners alike have been trying to find other faster methods.

MacMillan (1986) and Barone-Adesi and Whaley (1987) develop an approximate analytical formula for American options. Their approximation is very fast and many times faster than most other methods. A serious shortcoming of their method is that it is not very accurate, especially for long maturity options. Therefore the applicability of their method is quite limited. Another drawback of this analytical method is that it is not convergent because there is no control parameter to change to improve the accuracy.

Because lower bounds and upper bounds exist for American options, some researchers have attacked the problem of pricing American options from yet another angle. Generally they first prove a lower bound and an upper bound for the American option price. Then they adopt an interpolation scheme to price the American option. The accuracy of these methods depends crucially on the tightness of the bounds and the interpolation scheme. These methods include Johnson (1983) and Broadie and Detemple (1996). These methods can be quite fast, but an undesirable feature is that they all need regression coefficients which in turn require computing a large set of options accurately. Like the analytical approximation, methods based on interpolation schemes are not convergent.

A fourth group of methods includes those which are approximate schemes based on exact representations of the free boundary problem of pricing American options or the partial differential equation (PDE) satisfied by American option prices. This group includes Geske and Johnson (1984), Bunch and Johnson (1992), Huang, Subrahmanyam, and Yu (1996), and Carr (1997). These methods can be made as accurate as desired if more and more terms are included in the approximations. A serious problem is that, as more and more terms are included, the methods become more and more time consuming. A potential advantage of an exact representation of the American option problem is that it can be subjected to various approximation schemes. Geske and Johnson (1984) use a four-point Richardson extrapolation scheme to approximate their American option formula, which involves an infinite series of multivariate cumulative normal functions. Because multivariate cumulative normal functions require the computation of multidimensional integrals, the possibility to include more terms in their approximation to gain accuracy is very limited. Bunch and Johnson (1992) implement a modified two-point Geske-Johnson approach to avoid the multidimensional integrals. Carr (1997) discretizes only the time dimension of the evaluation PDE. Carr, Jarrow, and Myneni (1992), Jacka (1991), and Kim (1990) obtain formulas representing the early exercise premium of an American option as an integral (hereafter the integral representation method) which has the early exercise boundary in it. To avoid having to compute many early exercise points, Huang, Subrahmanyam, and Yu (1996) implement a four-point Richardson extrapolation scheme to the integral representation method. Only six points on an approximation to the early exercise boundary are needed. Since the integral representation method involves only the univariate cumulative normal function, their method is very fast, but it is not very accurate, especially for moderate and long maturity options, for example, LEAPS.[1]

In this article, we propose another approximation based on the integral representation method. The key insight of our approximation is that the early exercise boundary B_t appears only as an argument to the logarithmic function in the integral for the early exercise premium.[2] Therefore the integral does not depend on B_t critically. Consequently, we propose to approximate the early exercise boundary as a multipiece exponential function. Fortunately the resulting integral can then be evaluated in closed form. Because the most important time dependencies in the premium integral are integrated analytically, a very fast and accurate method is obtained.

The layout of the remainder of this article is as follows. We detail our approximation in Section 1. Numerical results are presented in Section 2. We demonstrate there that a three-point extrapolation scheme attains penny accuracy for a large set of options (1290). Our method has the advantage that it does not involve regression coefficients and is convergent. Our approximation also has several other useful features. It shares with many other methods that it is exact in the limit as the time to maturity goes to zero. It also shares the property with the lower and upper bound approximation (LUBA) of Broadie and Detemple (1996) that it is exact as the time to maturity goes to infinity, the perpetual case. Because our method is exact in both extreme cases, it is not very surprising that it yields good results for intermediate maturity options. On the other hand, as the time to maturity becomes longer, most of the other methods necessarily become less accurate. Section 3 summarizes and concludes the article.

1. Derivation and Implementation

Under the usual assumptions of constant riskless interest rate r, dividend yield δ and volatility σ, and a log-normal process for the underlying asset, Carr, Jarrow, and Myneni (1992), Jacka (1991), and Kim (1990) obtain the following formula for the price of an American put:[3]

$$P_A = P_E + \int_0^T \left[rKe^{-rt} N(-d_2(S, B_t, t)) - \delta S e^{-\delta t} N(-d_1(S, B_t, t)) \right] dt$$

[1] Long-term equity anticipation securities (LEAPS) are long-term exchange-traded options. They last up to 3 years.

[2] See Equation (1) in Section 1 of this article.

[3] We consider American puts only. American calls can be evaluated similarly or can be priced using the parity result of McDonald and Schroder (1990) for American options: $C(S, K, r, \delta, \sigma, T) = P(K, S, \delta, r, \sigma, T)$.

$$= P_E + K(1 - e^{-rT}) - S(1 - e^{-\delta T}) - K\int_0^T re^{-rt} N(d_2(S, B_t, t))\, dt$$
$$+ S\int_0^T \delta e^{-\delta t} N(d_1(S, B_t, t))\, dt, \qquad (1)$$

where

$$d_1(x, y, t) = \frac{\log(x/y) + (r - \delta + \sigma^2/2)t}{\sigma\sqrt{t}},$$
$$d_2(x, y, t) = d_1(x, y, t) - \sigma\sqrt{t},$$

and P_E is the price of the corresponding Black and Scholes (1973) European option formula. The early exercise boundary B_t solves the following integral equation:

$$K - B_t = P_E(B_t, K, T - t) + K(1 - e^{-r(T-t)})$$
$$- K\int_t^T re^{-r(s-t)} N(d_2(B_t, B_s, s - t))\, ds - B_t(1 - e^{-\delta(T-t)})$$
$$+ B_t\int_t^T \delta e^{-\delta(s-t)} N(d_1(B_t, B_s, s - t))\, ds. \qquad (2)$$

Once B_t is obtained, the price of the American put can be calculated easily. But solving for B_t is a very time-consuming process.

Huang, Subrahmanyam, and Yu (1996) use the Richardson extrapolation method to tackle this difficulty. They only calculate a few points on an approximation to the exercise boundary B_t. A sequence of approximate values of the option is obtained and then extrapolated to yield the value of P_A. To generate the approximate sequence, they evaluate the integrals in Equations (1) and (2) by approximating the integrands as step functions. This approximation amounts to the assumption that these integrands have no time dependence during each sub-interval. Approximating the integrands as step functions is obviously not very accurate. To improve accuracy, extrapolation is needed.

However, there is a special feature in Equation (1) which to our knowledge has not been utilized in the literature. Note that B_t appears only as $\log(S/B_t)$ in the definitions of $d_1(\cdot, \cdot, \cdot)$ and $d_2(\cdot, \cdot, \cdot)$. Therefore the integral in Equation (1) does not depend on the exact values of B_t critically. To make use of this property, instead of assuming everything to be a constant during each subinterval, we only assume that B_t is an exponential function during each subinterval. Fortunately the integrals in Equations (1) and (2) can then be evaluated in closed form. Since the integral does not depend on B_t critically and the other time dependencies are integrated analytically, our approximation results in a very simple, accurate, and fast method.

It should be pointed out that Omberg (1987) has also used an exponential early exercise boundary to price American options but in a different way. Because he does not use the optimal boundary, his method always underprices the options. On the other hand, we use the exponential boundary as a calculating device to evaluate the premium integral in an exact formulation. There is no inherent underpricing in our use of the exponential boundary. As a matter of fact, Broadie and Detemple (1996) prove that a certain approximate early exercise boundary in the integral representation method yields an upper bound for the option price. It should also be noted that a multipiece exponential boundary presents no problem in our application, but the first passage time problem in Omberg (1987) is probably impossible analytically.

Assume B_t to be an exponential function $B\exp(bt)$ for the interval $[t_1, t_2]$. Consider the integral

$$I_1 = \int_{t_1}^{t_2} r e^{-rt} N(d_2(S, Be^{bt}, t))dt. \tag{3}$$

Define $x_1 = (r - \delta - b - \sigma^2/2)/\sigma$, $x_2 = \log(S/B)/\sigma$. Then $d_2(S, B_t, t) = x_1 t^{1/2} + x_2 t^{-1/2}$. Integration by parts yields

$$\begin{aligned} I_1 &= -e^{-rt} N(x_1 t^{1/2} + x_2 t^{-1/2})\Big|_{t_1}^{t_2} \\ &\quad + \int_{t_1}^{t_2} e^{-rt} n(x_1 t^{1/2} + x_2 t^{-1/2}) \left(\frac{x_1}{2} t^{-1/2} - \frac{x_2}{2} t^{-3/2}\right) dt \\ &= e^{-rt_1} N(x_1 t_1^{1/2} + x_2 t_1^{-1/2}) - e^{-rt_2} N(x_1 t_2^{1/2} + x_2 t_2^{-1/2}) \\ &\quad + \frac{e^{-x_1 x_2}}{\sqrt{2\pi}} \int_{t_1}^{t_2} e^{-\frac{1}{2}((x_1^2 + 2r)t + x_2^2 t^{-1})} \left(\frac{x_1}{2} t^{-1/2} - \frac{x_2}{2} t^{-3/2}\right) dt \\ &= e^{-rt_1} N(x_1 t_1^{1/2} + x_2 t_1^{-1/2}) - e^{-rt_2} N(x_1 t_2^{1/2} + x_2 t_2^{-1/2}) \\ &\quad + \frac{e^{-x_1 x_2}}{\sqrt{2\pi}} \int_{t_1}^{t_2} e^{-\frac{1}{2}(x_3^2 t + x_2^2 t^{-1})} \left(\frac{x_1}{2} t^{-1/2} - \frac{x_2}{2} t^{-3/2}\right) dt, \end{aligned} \tag{4}$$

where $x_3 = \sqrt{x_1^2 + 2r}$. The above integral can be evaluated analytically by making use of the following identities:

$$dN(x_3 t^{1/2} + x_2 t^{-1/2}) = \frac{e^{-x_3 x_2}}{\sqrt{2\pi}} e^{-\frac{1}{2}(x_3^2 t + x_2^2 t^{-1})} \left(\frac{x_3}{2} t^{-1/2} - \frac{x_2}{2} t^{-3/2}\right),$$

$$dN(x_3 t^{1/2} - x_2 t^{-1/2}) = \frac{e^{-x_3 x_2}}{\sqrt{2\pi}} e^{-\frac{1}{2}(x_3^2 t + x_2^2 t^{-1})} \left(\frac{x_3}{2} t^{-1/2} + \frac{x_2}{2} t^{-3/2}\right).$$

It is now straightforward to show that

$$\begin{aligned} I_1 = {} & e^{-rt_1} N\left(x_1\sqrt{t_1} + \frac{x_2}{\sqrt{t_1}}\right) - e^{-rt_2} N\left(x_1\sqrt{t_2} + \frac{x_2}{\sqrt{t_2}}\right) \\ & + \frac{1}{2}\left(\frac{x_1}{x_3} + 1\right) e^{x_2(x_3 - x_1)} \\ & \times \left(N\left(x_3\sqrt{t_2} + \frac{x_2}{\sqrt{t_2}}\right) - N\left(x_3\sqrt{t_1} + \frac{x_2}{\sqrt{t_1}}\right)\right) \\ & + \frac{1}{2}\left(\frac{x_1}{x_3} - 1\right) e^{-x_2(x_3 + x_1)} \\ & \times \left(N\left(x_3\sqrt{t_2} - \frac{x_2}{\sqrt{t_2}}\right) - N\left(x_3\sqrt{t_1} - \frac{x_2}{\sqrt{t_1}}\right)\right). \end{aligned} \tag{5}$$

If we define $y_1 = (r - \delta - b + \sigma^2/2)/\sigma$, $y_2 = \log(S/B)/\sigma$, and $y_3 = \sqrt{y_1^2 + 2\delta}$, a similar derivation would yield

$$\begin{aligned} I_2 = {} & \int_{t_1}^{t_2} \delta e^{-\delta t} N(d_1(S, Be^{bt}, t))\, dt \\ = {} & e^{-rt_1} N\left(y_1\sqrt{t_1} + \frac{y_2}{\sqrt{t_1}}\right) - e^{-rt_2} N\left(y_1\sqrt{t_2} + \frac{y_2}{\sqrt{t_2}}\right) \\ & + \frac{1}{2}\left(\frac{y_1}{y_3} + 1\right) e^{y_2(y_3 - y_1)} \left(N\left(y_3\sqrt{t_2} + \frac{y_2}{\sqrt{t_2}}\right) - N\left(y_3\sqrt{t_1} + \frac{y_2}{\sqrt{t_1}}\right)\right) \\ & + \frac{1}{2}\left(\frac{y_1}{y_3} - 1\right) e^{-y_2(y_3 + y_1)} \\ & \times \left(N\left(y_3\sqrt{t_2} - \frac{y_2}{\sqrt{t_2}}\right) - N\left(y_3\sqrt{t_1} - \frac{y_2}{\sqrt{t_1}}\right)\right). \end{aligned} \tag{6}$$

Finally, if we define

$$\begin{aligned} & I(t_1, t_2, x, y, z, \phi, \nu) \\ & = e^{-\nu t_1} N\left(z_1\sqrt{t_1} + \frac{z_2}{\sqrt{t_1}}\right) - e^{-\nu t_2} N\left(z_1\sqrt{t_2} + \frac{z_2}{\sqrt{t_2}}\right) \\ & \quad + \frac{1}{2}\left(\frac{z_1}{z_3} + 1\right) e^{z_2(z_3 - z_1)} \left(N\left(z_3\sqrt{t_2} + \frac{z_2}{\sqrt{t_2}}\right) - N\left(z_3 t_1^{1/2} + \frac{z_2}{t_1^{1/2}}\right)\right) \\ & \quad + \frac{1}{2}\left(\frac{z_1}{z_3} - 1\right) e^{-z_2(z_3 + z_1)} \left(N\left(z_3\sqrt{t_2} - \frac{z_2}{\sqrt{t_2}}\right) - N\left(z_3\sqrt{t_1} - \frac{z_2}{\sqrt{t_1}}\right)\right), \end{aligned} \tag{7}$$

where

$$z_1 = \frac{r - \delta - z + \phi\sigma^2/2}{\sigma},$$
$$z_2 = \frac{\log(x/y)}{\sigma},$$
$$z_3 = \sqrt{z_1^2 + 2\nu},$$

then I_1 and I_2 can be expressed neatly as

$$I_1 = I(t_1, t_2, S, B, b, -1, r), \tag{8}$$
$$I_2 = I(t_1, t_2, S, B, b, 1, \delta). \tag{9}$$

If we define P_1, P_2, P_3, etc., as the approximate option values corresponding to approximating the early exercise boundary as a one-piece exponential function ($B_{11}e^{b_{11}t}$), a two-piece exponential function ($B_{22}e^{b_{22}t}$, $B_{21}e^{b_{21}t}$), and a three-piece exponential function ($B_{33}e^{b_{33}t}$, $B_{32}e^{b_{32}t}$, $B_{31}e^{b_{31}t}$), etc.,[4] then the P's are given by

$$P_1 = \begin{cases} P_E + K(1 - e^{-rT}) - S(1 - e^{-\delta T}) \\ -KI(0, T, S, B_{11}, b_{11}, -1, r) \\ +SI(0, T, S, B_{11}, b_{11}, 1, \delta) \quad \text{if } S > B_{11}, \\ K - S \quad \text{if } S \leq B_{11}. \end{cases} \tag{10}$$

$$P_2 = \begin{cases} P_E + K(1 - e^{-rT}) - S(1 - e^{-\delta T}) \\ -KI(0, T/2, S, B_{22}, b_{22}, -1, r) \\ +SI(0, T/2, S, B_{22}, b_{22}, 1, \delta) \\ -KI(T/2, T, S, B_{21}, b_{21}, -1, r) \\ +SI(T/2, T, S, B_{21}, b_{21}, 1, \delta) \quad \text{if } S > B_{22}, \\ K - S \quad \text{if } S \leq B_{22}. \end{cases} \tag{11}$$

$$P_3 = \begin{cases} P_E + K(1 - e^{-rT}) - S(1 - e^{-\delta T}) \\ -KI(0, T/3, S, B_{33}, b_{33}, -1, r) \\ +SI(0, T/3, S, B_{33}, b_{33}, 1, \delta) \\ -KI(T/3, 2T/3, S, B_{32}, b_{32}, -1, r) \\ +SI(T/3, 2T/3, S, B_{32}, b_{32}, 1, \delta) \\ -KI(2T/3, T, S, B_{31}, b_{31}, -1, r) \\ +SI(2T/3, T, S, B_{31}, b_{31}, 1, \delta) \quad \text{if } S > B_{33}, \\ K - S \quad \text{if } S \leq B_{33}. \end{cases} \tag{12}$$

Other P's follow similar patterns.

[4] The intervals are divided into equal subintervals. For example, if the early exercise boundary $\{B_t\}$ is approximated by a two-piece exponential function, then $B_t = B_{22}e^{b_{22}t}$ if $t \in [0, T/2)$, and $B_t = B_{21}e^{b_{21}t}$ if $t \in [T/2, T]$.

To determine the B's and b's, we apply the "value-match" and "high-contact" conditions. For example, to determine B_{21} and b_{21}, applying the value-match and high-contact conditions at $t = T/2$ yields

$$\begin{aligned} K - B_{21}e^{b_{21}T/2} = \; & P_E(B_{21}e^{b_{21}T/2}, K, T/2) + K(1 - e^{-rT/2}) \\ & - B_{21}e^{b_{21}T/2}(1 - e^{-\delta T/2}) \\ & - KI(0, T/2, B_{21}e^{b_{21}T/2}, B_{21}e^{b_{21}T/2}, b_{21}, -1, r) \\ & + B_{21}e^{b_{21}T/2}I(0, T/2, B_{21}e^{b_{21}T/2}, B_{21}e^{b_{21}T/2}, b_{21}, 1, \delta), \end{aligned} \tag{13}$$

$$\begin{aligned} -1 = \; & -e^{-\delta T/2}N(-d_1(B_{21}e^{b_{21}T/2}, K, T/2)) - (1 - e^{-\delta T/2}) \\ & - KI_S(0, T/2, B_{21}e^{b_{21}T/2}, B_{21}e^{b_{21}T/2}, b_{21}, -1, r) \\ & + I(0, T/2, B_{21}e^{b_{21}T/2}, B_{21}e^{b_{21}T/2}, b_{21}, 1, \delta) \\ & + B_{21}e^{b_{21}T/2}I_S(0, T/2, B_{21}e^{b_{21}T/2}, B_{21}e^{b_{21}T/2}, b_{21}, 1, \delta). \end{aligned} \tag{14}$$

Similarly, to determine B_{22} and b_{22}, applying the value-match and high-contact conditions at $t = 0$ yields

$$\begin{aligned} K - B_{22} = \; & P_E(B_{22}, K, T) + K(1 - e^{-rT}) - B_{22}(1 - e^{-\delta T}) \\ & - KI(0, T/2, B_{22}, B_{22}, b_{22}, -1, r) \\ & + B_{22}I(0, T/2, B_{22}, B_{22}, b_{22}, 1, \delta) \\ & - KI(T/2, T, B_{22}, B_{21}, b_{21}, -1, r) \\ & + B_{22}I(T/2, T, B_{22}, B_{21}, b_{21}, 1, \delta), \end{aligned} \tag{15}$$

$$\begin{aligned} -1 = \; & -e^{-\delta T}N(-d_1(B_{22}, K, T)) - (1 - e^{-\delta T}) \\ & - KI_S(0, T/2, B_{22}, B_{22}, b_{22}, -1, r) \\ & + I(0, T/2, B_{22}, B_{22}, b_{22}, 1, \delta) \\ & + B_{22}I_S(0, T/2, B_{22}, B_{22}, b_{22}, 1, \delta) \\ & - KI_S(T/2, T, B_{22}, B_{21}, b_{21}, 1, \delta) \\ & + I(T/2, T, B_{22}, B_{21}, b_{21}, 1, \delta) \\ & + B_{22}I_S(T/2, T, B_{22}, B21, b_{21}, 1, \delta). \end{aligned} \tag{16}$$

The function $I_S(\cdots)$ is defined by

$$\begin{aligned} & I_S(t_1, t_2, S, B, b, \phi, \nu) \\ & = \frac{\partial I}{\partial S} = \left(\frac{e^{-\nu t_1}}{\sqrt{t_1}} n\left(z_1\sqrt{t_1} + \frac{z_2}{\sqrt{t_1}} \right) - \frac{e^{-\nu t_2}}{\sqrt{t_2}} n\left(z_1\sqrt{t_2} + \frac{z_2}{\sqrt{t_2}} \right) \right) \frac{1}{\sigma S} \\ & \quad + \frac{1}{2\sigma S} e^{z_2(z_3 - z_1)} \left(N\left(z_3\sqrt{t_2} + \frac{z_2}{\sqrt{t_2}} \right) - N\left(z_3 t_1^{1/2} + \frac{z_2}{t_1^{1/2}} \right) \right) (z_3 - z_1) \end{aligned}$$

$$
\begin{aligned}
&+ \frac{1}{2\sigma S} e^{z_2(z_3-z_1)} \left(n\left(z_3\sqrt{t_2} + \frac{z_2}{\sqrt{t_2}} \right) \frac{1}{\sqrt{t_2}} \right. \\
&\qquad\qquad\qquad \left. - n\left(z_3 t_1^{1/2} + \frac{z_2}{t_1^{1/2}} \right) \right) \frac{1}{\sqrt{t_1}} \Bigg) \Bigg) \\
&- \frac{1}{2\sigma S} e^{-z_2(z_3+z_1)} \left(N\left(z_3\sqrt{t_2} + \frac{z_2}{\sqrt{t_2}} \right) \right. \\
&\qquad\qquad\qquad \left. - N\left(z_3 t_1^{1/2} + \frac{z_2}{t_1^{1/2}} \right) \right) (z_3 + z_1) \\
&- \frac{1}{2\sigma S} e^{-z_2(z_3+z_1)} \left(n\left(z_3\sqrt{t_2} + \frac{z_2}{\sqrt{t_2}} \right) \frac{1}{\sqrt{t_2}} \right. \\
&\qquad\qquad\qquad \left. - n\left(z_3 t_1^{1/2} + \frac{z_2}{t_1^{1/2}} \right) \right) \frac{1}{\sqrt{t_1}} \Bigg) \Bigg). \qquad (17)
\end{aligned}
$$

The B's and b's can be easily and quickly obtained using the two-dimensional Newton-Raphson method [for a good reference, see Press et al. (1992)]. To find B_{11} and b_{11}, the approximation of MacMillan (1986) and Barone-Adesi and Whaley (1987) provides a good initial estimate for B_{11}, and 0 provides a good initial estimate for b_{11}. Once B_{11} and b_{11} are found, they provide good initial estimates for finding B_{21} and b_{21}, respectively. To find B_{22} and b_{22}, B_{21} and b_{21} are good initial estimates.

For readers who may want to implement this method, some numerical numbers may help. Take, for example, the first option in Table 2. Estimate of B_{11} by the approximation of MacMillan (1986) and Barone-Adesi and Whaley (1987) is 52.452. Using 52.452 and 0 as the initial estimates for B_{11} and b_{11}, they are found to be 54.457 and 0.036, respectively. Using 54.457 and 0.036 as the initial estimates for B_{21} and b_{21}, they are found to be 52.389 and 0.061, respectively. Finally, using 52.389 and 0.061 as the initial estimates for B_{22} and b_{22}, they are found to be 54.453 and 0.0307, respectively. The corresponding values of B_{31} and b_{31}, B_{32} and b_{32}, and B_{33} and b_{33} are found to be 50.868 and 0.075, 53.705 and 0.045, 54.452 and 0.029, respectively. Once the B's and b's are obtained, P_1, P_2, and P_3 can be easily obtained using Equations (10), (11), and (12). For this particular example, it is noted that all the initial estimates are close to their convergent values. Therefore, few iterations of the Newton-Raphson method are needed to obtain each pair of the B's and b's. This behavior is generally true and is the reason that our method is very fast.

In cases where the difference between the exercise boundary at maturity, which is $B_T = \min(K, Kr/\delta)$ for an American put, and the initial estimate for B_{11} using the MacMillan (1986) and Barone-Adesi and Whaley (1987) approximation is smaller than 5% of B_T, the early exercise boundary will be very flat. In this case, we recommend replacing the multipiece exponential functions by the multipiece constants (a step function). That is, in the Newton-Raphson search, the exponents are initialized to zero and not updated. This is to ensure the convergence of the Newton-Raphson method in the extreme case that the early exercise boundary is very flat. Five percent is only suggestive. We have tested other smaller numbers, like 1%. They have all worked well in our test.

In this article, we recommend a three-point Richardson scheme to price American options. If P_1, P_2, and P_3 are the values given in Equations (10), (11), and (12), then the American put price is approximated by

$$\hat{P}_A = 4.5P_3 - 4P_2 + 0.5P_1.$$

We demonstrate in the following section that this results in a very accurate and fast method for pricing American options. In fact, we also demonstrate that even the unextrapolated values of P_1, P_2, and P_3 are very good approximate values for the true American option prices. This of course confirms our argument that the true values do not depend on the exact values of the early exercise boundary critically. To show the rate of convergence, the two-point and four-point Richardson schemes are also included in the large sample test in Table 3.

Before we leave this section, we point out that our method applies equally well to other nonstandard American options when the early exercise premium can be represented as integrals involving the cumulative normal function. One particular example is the American barrier option. In the setting of this section, Gao, Huang, and Subrahmanyam (1996) have derived analytical formulas for American barrier options. For example, when the dividend yield is zero, the price of an American "up-and-out" put option is given by

$$\begin{aligned} P_A^{uo} \;=\; & P_E^{uo} + \int_0^T e^{-rt} rK[N(-d_2(S, B_t, t)) \\ & - (H/S)^{2\lambda-2} N(-d_2(H^2/S, B_t, t))]dt, \end{aligned} \tag{18}$$

where

$$P_E^{uo} = P_E(S, K) - (H/S)^{2\lambda-2} P_E(H^2/S, K)$$

is the price of a European "up-and-out" put option, and P_E is the Black-Scholes formula for standard European put options, H is the barrier, and $\lambda = (r + \sigma^2/2)/\sigma^2$. The form of the above formula is very similar to Equation (1), and therefore our method of approximating the early exercise boundary as a multipiece exponential function easily applies. Gao, Huang, and Subrahmanyam (1996) have also derived formulas when the dividend yield is not zero and for other types of American barrier options. But all those formulas have forms similar to that of Equation (18), therefore, our method applies to the nonzero dividend case and other types of American barrier options too.

2. Numerical Results and Discussions

In this section we report numerical results to demonstrate the speed and accuracy of our approximation scheme. We compare our two-point, three-point, and four-point multipiece exponential function methods (hereafter EXP2, EXP3, and EXP4) with an 800-time-step binomial tree model (hereafter BT800), the four-point extrapolation scheme of Geske and Johnson (1984) (hereafter GJ4), the modified two-point Geske and Johnson method of Bunch and Johnson (1992) (hereafter MGJ2), the four-point and six-point recursive methods of Huang, Subrahmanyam, and Yu (1996) (hereafter HSY4 and HSY6), the four-point and six-point randomization methods of Carr (1997) (hereafter RAN4 and RAN6), and the lower and upper bound approximation of Broadie and Detemple (1996) (hereafter LUBA). In Table 3 we also compare the binomial Black and Scholes method with Richardson extrapolation of Broadie and Detemple (1996) with $N = 25$ and $N = 50$ (hereafter BBSR25 and BBSR50).[5] BBSR25 means that the price is approximated by $2P_2 - P_1$, where P_1 and P_2 are computed using the BBS method of Broadie and Detemple (1996) with $N = 25$ and $N = 50$, respectively. BBSR50 has a similar interpretation.

To assess the accuracies of these methods, we choose a binomial model with $N = 10{,}000$ time steps as our benchmark for the true values of the options considered. We use the root of mean squared errors (RMSE) to measure the overall accuracy of the options in Tables 1 and 2. More error measures are included in the large sample test in Table 3. The computational speed is measured using the total CPU time (in seconds) required to price the whole set of the options considered. We follow Bunch and Johnson (1992) in the implementation of MGJ2. A more sophisticated optimization routine is not likely to affect the speed and accuracy significantly. GJ4 is not included in Table 3 for the large sample test since it is too slow. However, it is reasonable to assume that it is much slower than MGJ2.

To test the accuracies of these methods for short and moderate maturity options, Table 1 reports the results for the 20 options in Table 1 in Broadie and Detemple (1996). To test the accuracies of these methods for longer maturity options, Table 2 reports the results for the 20 options in Table V in Barone-Adesi and Whaley (1987).[6] From these tables it is clear that the

[5] BBSR25 and BBSR50 are not included in other tables due to space limitation.

[6] The first five options in Table 2 have parameters beyond the optimal parameter range for LUBA. Nevertheless, LUBA has priced these five options accurately. Inclusion of these five options should not have affected RMSE of LUBA. This comment also applies to Table 5.

Table 1
Prices of American call options (K = \$100, T = 0.5 years)

(1) (S, σ, r, δ)	(2) TRUE	(3) BT800	(4) GJ4	(5) MGJ2	(6) HSY4	(7) HSY6	(8) LUBA	(9) RAN4	(10) RAN6	(11) EXP3
(80, 0.2, 0.03, 0.07)	0.2194	0.2194	0.2191	0.2186	0.2199	0.2197	0.2195	0.2188	0.2188	0.2196
(90, 0.2, 0.03, 0.07)	1.3864	1.3874	1.3849	1.3818	1.3898	1.3895	1.3862	1.3802	1.3844	1.3872
(100, 0.2, 0.03, 0.07)	4.7825	4.7818	4.7851	4.7862	4.8044	4.7804	4.7821	4.7728	4.7794	4.7837
(110, 0.2, 0.03, 0.07)	11.0978	11.0986	11.0889	11.2553	11.0686	11.0983	11.0976	11.0893	11.0957	11.0993
(120, 0.2, 0.03, 0.07)	20.0004	20.0000	20.0073	20.0000	20.0531	20.0006	20.0000	20.0000	20.0000	20.0005
(80, 0.4, 0.03, 0.07)	2.6889	2.6887	2.6864	2.6827	2.6897	2.6909	2.6893	2.6787	2.6857	2.6899
(90, 0.4, 0.03, 0.07)	5.7223	5.7238	5.7212	5.7163	5.7361	5.7286	5.7231	5.7113	5.7168	5.7237
(100, 0.4, 0.03, 0.07)	10.2385	10.2365	10.2451	10.2351	10.2752	10.2372	10.2402	10.2205	10.2323	10.2404
(110, 0.4, 0.03, 0.07)	16.1812	16.1828	16.1831	16.2107	16.2012	16.1854	16.1817	16.1629	16.1745	16.1831
(120, 0.4, 0.03, 0.07)	23.3598	23.3597	23.3419	23.4771	23.3288	23.3850	23.3574	23.3389	23.3542	23.3622
(80, 0.3, 0.00, 0.07)	1.0373	1.0371	1.0351	1.0317	1.0374	1.0388	1.0373	1.0297	1.0352	1.0381
(90, 0.3, 0.00, 0.07)	3.1233	3.1222	3.1240	3.1146	3.1438	3.1282	3.1232	3.1139	3.1193	3.1247
(100, 0.3, 0.00, 0.07)	7.0354	7.0343	7.0375	7.0413	7.0667	7.0332	7.0355	7.0210	7.0307	7.0371
(110, 0.3, 0.00, 0.07)	12.9552	12.9568	12.9339	13.0637	12.9091	12.9759	12.9531	12.9437	12.9509	12.9574
(120, 0.3, 0.00, 0.07)	20.7173	20.7147	20.7423	20.4380	20.7268	20.6969	20.7208	20.7056	20.7154	20.7194
(80, 0.3, 0.07, 0.03)	1.6644	1.6636	1.6644	1.6644	1.6644	1.6644	1.6644	1.6604	1.6644	1.6644
(90, 0.3, 0.07, 0.03)	4.4947	4.4926	4.4946	4.4947	4.4947	4.4947	4.4947	4.4959	4.4947	4.4947
(100, 0.3, 0.07, 0.03)	9.2504	9.2481	9.2509	9.2506	9.2506	9.2506	9.2506	9.2513	9.2506	9.2506
(110, 0.3, 0.07, 0.03)	15.7977	15.7993	15.7973	15.7975	15.7975	15.7975	15.7975	15.7994	15.7975	15.7975
(120, 0.3,0.07, 0.03)	23.7061	23.7059	23.7082	23.7062	23.7062	23.7062	23.7062	23.7027	23.7065	23.7062
RMSE		0.0012	0.0090	0.0805	0.0231	0.0089	0.0012	0.0104	0.0035	0.0013

The "TRUE" value is based on the binomial tree model with $N = 10,000$ time steps. Columns 3–11 represent the binomial tree model with $N = 800$ time steps, the four-point method of Geske and Johnson (1984), the modified two-point Geske-Johnson method of Bunch and Johnson (1992), the four-point and six-point recursive schemes of Huang, Subrahmanyam, and Yu (1996), the lower and upper bound approximation of Broadie and Detemple (1996), the four-point and six-point randomization methods of Carr (1997), and the three-point multipiece exponential boundary method of this article, respectively. RMSE is the root of mean squared errors.

Table 2
Prices of American put options (K = \$100, T = 3.0 years, $\sigma = 0.2$, r = 0.08)

(1) (S, δ)	(2) TRUE	(3) BT800	(4) GJ4	(5) MGJ2	(6) HSY4	(7) HSY6	(8) LUBA	(9) RAN4	(10) RAN6	(11) EXP3
(80, 0.12)	25.6577	25.6564	25.6529	25.9487	25.6862	25.7021	25.6568	25.6542	25.6577	25.6570
(90, 0.12)	20.0832	20.0839	20.1089	20.2009	20.1275	20.0905	20.0834	20.0838	20.0830	20.0817
(100, 0.12)	15.4981	15.4945	15.5122	15.5495	15.5356	15.5020	15.4985	15.4977	15.4984	15.4970
(110, 0.12)	11.8032	11.8013	11.8023	11.8236	11.8228	11.8121	11.8032	11.8047	11.8032	11.8022
(120, 0.12)	8.8856	8.8873	8.8803	8.8965	8.8937	8.8921	8.8855	8.8850	8.8856	8.8850
(80, 0.08)	22.2050	22.2043	22.2079	22.7106	22.2445	22.1493	22.1985	22.1969	22.2032	22.2084
(90, 0.08)	16.2071	16.2084	16.1639	16.5305	16.1340	16.2578	16.1986	16.1967	16.2034	16.2106
(100, 0.08)	11.7037	11.7013	11.7053	11.8106	11.7175	11.7237	11.6988	11.6882	11.6994	11.7066
(110, 0.08)	8.3671	8.3664	8.3886	8.4072	8.4355	8.3563	8.3630	8.3537	8.3621	8.3695
(120, 0.08)	5.9299	5.9322	5.9435	5.9310	5.9881	5.9323	5.9261	5.9178	5.9252	5.9323
(80, 0.04)	20.3500	20.3489	20.5134	20.0000	20.5225	20.3932	20.3335	20.3499	20.3492	20.3511
(90, 0.04)	13.4968	13.4969	13.5246	14.0246	13.3784	13.4602	13.4982	13.4961	13.4939	13.5000
(100, 0.04)	8.9438	8.9423	8.8414	9.1086	8.8038	8.9891	8.9424	8.9357	8.9403	8.9474
(110, 0.04)	5.9119	5.9115	5.8904	5.9310	5.9186	5.9269	5.9122	5.9021	5.9077	5.9146
(120, 0.04)	3.8975	3.8988	3.9046	3.8823	3.9778	3.8834	3.8980	3.8870	3.8932	3.8997
(80, 0.0)	20.0000	20.0000	19.7313	20.0000	19.8458	19.9484	20.0000	20.0000	20.0000	20.0000
(90, 0.0)	11.6974	11.6955	11.8842	10.1758	11.7606	11.7047	11.6953	11.7011	11.6960	11.6991
(100, 0.0)	6.9320	6.9301	6.9266	6.9394	6.7859	6.9109	6.9346	6.9316	6.9301	6.9346
(110, 0.0)	4.1550	4.1539	4.1033	4.1453	4.0902	4.1897	4.1550	4.1531	4.1528	4.1571
(120, 0.0)	2.5102	2.5103	2.4906	2.4546	2.5591	2.5150	2.5110	2.5068	2.5082	2.5119
RMSE		0.0016	0.0872	0.4014	0.0853	0.0304	0.0048	0.0073	0.0028	0.0023

The "TRUE" value is based on the binomial tree model with $N = 10,000$ time steps. Columns 3–11 represent the binomial tree model with $N = 800$ time steps, the four-point method of Geske and Johnson (1984), the modified two-point Geske-Johnson method of Bunch and Johnson (1992), the four-point and six-point recursive schemes of Huang, Subrahmanyam, and Yu (1996), the lower and upper bound approximation of Broadie and Detemple (1996), the four-point and six-point randomization methods of Carr (1997), and the three-point multipiece exponential boundary method of this article, respectively. RMSE is the root of mean squared errors.

Table 3
Summary of 1250 randomly generated American puts

(1)	(2) BT800	(3) MGJ2	(4) HSY4	(5) HSY6	(6) LUBA	(7) RAN4	(8) RAN6	(9) BBSR25	(10) BBSR50	(11) EXP2	(12) EXP3	(13) EXP4
RMSE	0.0017	0.1255	0.0382	0.0147	0.0016	0.0101	0.0036	0.0054	0.0022	0.0115	0.0017	0.0004
MAE	0.0097	1.4775	0.3329	0.1390	0.0190	0.0527	0.0179	0.1037	0.0499	0.0613	0.0096	0.0027
AE > 0.01	0.0000	0.4432	0.5144	0.2816	0.0120	0.3928	0.1032	0.1072	0.0136	0.3672	0.0000	0.0000
AE > 99%	0.0071	0.9108	0.2217	0.0981	0.0108	0.0436	0.0152	0.0356	0.0136	0.0530	0.0076	0.0022
RMSRE	0.0148	0.5382	0.3204	0.0844	0.0102	0.0972	0.0308	0.0428	0.0170	0.0746	0.0142	0.0054
MARE	0.1430	5.5587	4.8770	1.1333	0.1871	1.0465	0.2546	0.6530	0.3243	0.5612	0.1483	0.0742
ARE > 99%	0.0762	3.6483	2.2173	0.4854	0.0549	0.6406	0.1677	0.3514	0.1293	0.4135	0.0883	0.0401
CPU (sec)	588.75	4.83	2.29	6.25	4.34	3.62	5.42	5.35	15.33	2.35	4.53	8.33

Columns 2–13 represent the binomial tree model with $N = 800$ time steps, modified two-point Geske-Johnson method of Bunch and Johnson (1992), the four-point and six-point recursive schemes of Huang, Subrahmanyam, and Yu (1996), the lower and upper bound approximation of Broadie and Detemple (1996), the four-point and six-point randomization methods of Carr (1997), the binomial Black and Scholes method with Richardson extrapolation of Broadie and Detemple (1996) with $N = 25$ and $N = 50$, and the two-point, three-point, and four-point multipiece exponential boundary method of this article, respectively. The first row represents the chosen methods. The second row is the root of mean squared errors (RMSE). The third row is the maximum absolute error (MAE). The fourth row is the fraction of options with absolute error greater than 1 cent (AE $>$ 0.01). The fifth row is the 99th percentile of the absolute error (AE $>$ 99%). Rows 6–8 are the percentage of the root of mean squared relative errors (RMSRE), the maximum absolute relative error (MARE) and the 99th percentile of the absolute relative error (ARE $>$ 99%), respectively. For example, 0.1430 in column 2 means the MARE is 0.1430%. Only options with prices greater than $0.5 are included in computing the error measures in rows 6-8. The last row reports the CPU time (seconds) for pricing the whole set of options on a Sparc-20 in FORTRAN.

performance of EXP3 is truly amazing. Its RMSE is only 0.0013 cents in Table 1, and 0.0023 cents in Table 2.

In order to have a more complete assessment concerning the accuracy and speed of these methods except GJ4, which is too slow, and BBSR25, BBSR50, EXP2, and EXP4, Table 3 reports the results from 1250 randomly generated American put options. The parameters are generated as follows: volatility σ is uniform between 0.1 and 0.6; time to maturity T is uniform between 0.0 and 3.0 years; the strike price is fixed at $K = 100.0$; current stock price S is uniform between 70.0 and 130.0; both the riskless interest rate and the dividend yield are uniform between 0.0 and 0.1. These values represent a wide range of parameter values. The maximum error of EXP3 is not only less than 1 cent for the 40 options in Tables 1 and 2, it is also less than 1 cent for the 1250 options in Table 3. BT800 and EXP4 are the only other methods whose maximum error is less than 1 cent for the 1250 options considered in Table 3. All other methods have multiple occurrences with pricing errors equal to or greater than 1 cent. Table 3 also shows that the multipiece exponential boundary method converges very quickly. Even though EXP4 is included in Table 3, it seems that it may never be needed since EXP3 is already an extremely accurate method. On the other hand, EXP2 may be acceptable in many applications. Nevertheless, since EXP3 is so fast that it can be used in most situations. Even though BBSR25 and BBSR50 are quite simple, they are quite accurate and fast. When the requirement of accuracy is not too stringent, BBSR could be the choice of method in many applications because it is simple, fast, and easy to program. In light of the tests in Tables 1, 2, and 3, EXP3, LUBA, and RAN6 are all extremely fast and accurate methods for pricing American options.

Overall, BT800, EXP3, LUBA, and RAN6 appear to have about the same accuracies. Obviously MGJ2 is the least accurate method. Even for the relatively short maturity options in Table 1, the pricing errors of MGJ2 are quite substantial. HSY4 and HSY6 appear to be dominated by EXP3, LUBA, and RAN6 in terms of speed and accuracy.

To understand better why EXP3 works so well, Table 4 considers the convergent behavior of the unextrapolated values of the approximate option prices. From the table, it is not difficult to see the reason. Even the unextrapolated values EXP-P1, EXP-P2, and EXP-P3 are quite close to the true option prices. Specifically, the one-point unextrapolated value EXP-P1 is much more accurate than the six-point unextrapolated values HSY-P6 and RAN-P6. As a matter of fact, EXP-P1 is only slightly less accurate than the six-point extrapolated value HSY6 for this set of options. Even for these long maturity options, the unextrapolated values EXP-P1, EXP-P2, and EXP-P3 are quite acceptable for many applications. Again, this behavior confirms the observation that the early exercise premium does not depend on the early exercise boundary critically. Since the other time dependencies are integrated analytically, very good approximate values are obtained. This is the key insight of this article. On the other hand, even the six-point unextrapolated values HSY-P6 and RAN-P6 have huge pricing errors.

Table 5 considers the accuracies of these methods for calculating the hedging ratio Δ. To compute Δ using the binomial tree model, we use the extended tree method described in Pelsser and Vorst (1994). Even though it is possible to derive analytical formulas of the Δ for the other methods, a numerical derivative is used. We point out that this does not increase the computing time significantly because the approximate early exercise boundary does not change when the stock price changes. Accuracy of the numerical derivative is checked by changing the step size and found to be very accurate. The general behavior concerning the prices also holds here for the hedging ratios. BT800, EXP3, LUBA and RAN6 all provide accurate hedging ratios. It should be noted that even though LUBA, RAN6, and EXP3 have similar accuracies for option prices, the RMSE of the hedging ratios of LUBA in Table 5 is more than 10 times that of RAN6 and EXP3. The reason is probably that LUBA is optimized for the best performance for option prices, not for hedging ratios. This points out a potential disadvantage of methods based on regression techniques.

To have some indication how the multipiece exponential boundary compares with the approximation to the true early exercise boundary, Figure 1 shows the one-piece, two-piece, and three-piece exponential boundaries and the approximate early exercise boundary. Parameters in Figure 1 correspond to options 11–15 in Table 2. Overall the multipiece exponential boundaries are not very close to the approximation boundary. Nevertheless, Table 2 reveals that EXP3 has priced these options quite accurately. Therefore this confirms our intuition that a very accurate estimate of the early exercise boundary is not required for pricing an American option accurately. For the parameters used in Figure 1, the multipiece exponential boundary is quite linear during each subinterval. Nevertheless, a multipiece linear boundary is not recommended because the resulting early exercise premium integral cannot be evaluated analytically.

3. Concluding Remarks

We have proposed a fast and accurate approximation for pricing American options. Our approximation is based on the observation that the early exercise premium does not depend on the exact values of the early exercise boundary critically. This insight allows us to approximate the early exercise boundary as a multipiece exponential function. Because the resulting integral of the early exercise premium can be evaluated analytically, a fast and accurate approximation is obtained. Numerical results show that our approximation based on a three-point extrapolation scheme has the accuracy of an 800-time-step binomial tree model, but is about 130 times faster.

Table 4
Unextrapolated put values (K = \$100, T = 3.0 years, $\sigma = 0.2$, r = 0.08)

(1) (S, δ)	(2) TRUE	(3) EXP-P1	(4) EXP-P2	(5) EXP-P3	(6) HSY-P2	(7) HSY-P4	(8) EXP-P6	(9) RAN-P3	(10) RAN-P4	(11) RAN-P6
(80, 0.12)	25.6577	25.6404	25.6543	25.6564	25.3230	25.5499	25.6087	24.9528	25.2663	25.3867
(90, 0.12)	20.0832	20.0679	20.0805	20.0821	19.8723	20.0010	20.0429	18.9587	19.4841	19.6752
(100, 0.12)	15.4981	15.4867	15.4964	15.4976	15.3623	15.4412	15.4688	14.1383	14.7863	15.0161
(110, 0.12)	11.8032	11.7949	11.8017	11.8026	11.7170	11.7655	11.7827	10.5027	11.1097	11.3309
(120, 0.12)	8.8856	8.8799	8.8844	8.8850	8.8324	8.8612	8.8718	7.8534	8.3163	8.4930
(80, 0.08)	22.2050	22.1650	22.1916	22.1983	21.5703	22.2163	22.2967	21.7577	21.9402	22.0170
(90, 0.08)	16.2071	16.1473	16.1882	16.1977	15.9365	16.2288	16.2538	15.3822	15.7510	15.8907
(100, 0.08)	11.7037	11.6417	11.6840	11.6938	11.5394	11.6880	11.7147	10.6768	11.1544	11.3263
(110, 0.08)	8.3671	8.3122	8.3488	8.3574	8.2370	8.3353	8.3623	7.3980	7.8407	8.0038
(120, 0.08)	5.9299	5.8857	5.9142	5.9214	5.8224	5.8971	5.9192	5.1787	5.5070	5.6342
(80, 0.04)	20.3500	20.3379	20.3447	20.3469	18.8206	20.1777	20.4122	20.1814	20.2493	20.2787
(90, 0.04)	13.4968	13.4459	13.4781	13.4866	13.1947	13.7667	13.7612	12.9199	13.1819	13.2803
(100, 0.04)	8.9438	8.8747	8.9197	8.9308	8.9527	9.1426	9.1053	8.1677	8.5315	8.6617
(110, 0.04)	5.9119	5.8435	5.8876	5.8985	5.9368	6.0075	5.9956	5.1826	5.5153	5.6381
(120, 0.04)	3.8975	3.8394	3.8761	3.8854	3.8829	3.9342	3.9389	3.3418	3.5822	3.6762
(80, 0.0)	20.0000	20.0000	20.0000	20.0000	16.6525	18.7463	19.2411	20.0000	20.0000	20.0000
(90, 0.0)	11.6974	11.6729	11.6878	11.6919	11.1602	12.0365	12.0819	11.3607	11.5205	11.5781
(100, 0.0)	6.9320	6.8832	6.9145	6.9225	7.1356	7.3556	7.2640	6.3889	6.6529	6.7441
(110, 0.0)	4.1550	4.1020	4.1362	4.1447	4.4050	4.4077	4.3392	3.6412	3.8815	3.9684
(120, 0.0)	2.5102	2.4646	2.4938	2.5010	2.6586	2.6376	2.6079	2.1269	2.2954	2.3607
RMSE		0.0437	0.0151	0.0081	0.8593	0.3260	0.2239	0.7883	0.4258	0.2923

The "TRUE" value is based on the binomial tree model with $N = 10,000$ time steps. Columns 3–5 represent the one-point, two-point, and three-point unextrapolated values of the multipiece exponential boundary method of this article, respectively. Columns 6–8 denote the two-point, four-point, and six-point unextrapolated values of the recursive method of Huang, Subrahmanyam, and Yu (1996), respectively. Columns 9–11 are the two-point, four-point, and six-point unextrapolated values of the randomization method of Carr (1997), respectively. RMSE is the root of mean squared errors.

Table 5
Hedge ratios (Δ's) of American puts ($K = 100$, $T = 3.0$ years, $\sigma = 0.2$, $r = 0.08$)

(1) (S, δ)	(2) TRUE	(3) BT800	(4) GJ4	(5) MGJ2	(6) HSY4	(7) HSY6	(8) LUBA	(9) RAN4	(10) RAN6	(11) EXP3
(80, 0.12)	−0.6103	−0.6101	−0.6046	−0.6367	−0.6094	−0.6125	−0.6101	−0.6096	−0.6104	−0.6104
(90, 0.12)	−0.5063	−0.5061	−0.5061	−0.5147	−0.5057	−0.5088	−0.5063	−0.5064	−0.5063	−0.5064
(100, 0.12)	−0.4122	−0.4122	−0.4141	−0.4172	−0.4139	−0.4115	−0.4123	−0.4122	−0.4122	−0.4122
(110, 0.12)	−0.3287	−0.3286	−0.3296	−0.3298	−0.3302	−0.3286	−0.3287	−0.3284	−0.3287	−0.3286
(120, 0.12)	−0.2569	−0.2569	−0.2569	−0.2577	−0.2576	−0.2573	−0.2569	−0.2576	−0.2569	−0.2569
(80, 0.08)	−0.6878	−0.6877	−0.7013	−0.3540	−0.7162	−0.6788	−0.6887	−0.6877	−0.6881	−0.6877
(90, 0.08)	−0.5189	−0.5187	−0.5161	−0.5439	−0.5152	−0.5150	−0.5186	−0.5196	−0.5190	−0.5190
(100, 0.08)	−0.3871	−0.3871	−0.3831	−0.4043	−0.3779	−0.3929	−0.3869	−0.3872	−0.3872	−0.3872
(110, 0.08)	−0.2847	−0.2847	−0.2845	−0.2897	−0.2831	−0.2847	−0.2846	−0.2842	−0.2847	−0.2847
(120, 0.08)	−0.2064	−0.2065	−0.2077	−0.2092	−0.2091	−0.2048	−0.2064	−0.2066	−0.2063	−0.2064
(80, 0.04)	−0.8374	−0.8375	−0.8268	−1.0000	−0.8507	−0.8500	−0.8338	−0.8368	−0.8377	−0.8372
(90, 0.04)	−0.5541	−0.5542	−0.5717	−0.5939	−0.5754	−0.5472	−0.5547	−0.5548	−0.5542	−0.5540
(100, 0.04)	−0.3691	−0.3692	−0.3729	−0.3853	−0.3568	−0.3664	−0.3689	−0.3696	−0.3691	−0.3691
(110, 0.04)	−0.2456	−0.2457	−0.2414	−0.2571	−0.2328	−0.2508	−0.2455	−0.2455	−0.2456	−0.2456
(120, 0.04)	−0.1628	−0.1630	−0.1615	−0.1645	−0.1606	−0.1630	−0.1628	−0.1631	−0.1628	−0.1628
(80, 0.0)	−1.0000	−1.0000	−0.9236	−1.0000	−0.9472	−0.9755	−1.0000	−1.0000	−1.0000	−1.0000
(90, 0.0)	−0.6209	−0.6216	−0.6352	−0.2222	−0.6506	−0.6419	−0.6173	−0.6211	−0.6209	−0.6207
(100, 0.0)	−0.3583	−0.3587	−0.3716	−0.3405	−0.3620	−0.3489	−0.3588	−0.3586	−0.3583	−0.3582
(110, 0.0)	−0.2109	−0.2112	−0.2092	−0.2160	−0.1969	−0.2112	−0.2108	−0.2109	−0.2109	−0.2109
(120, 0.0)	−0.1257	−0.1259	−0.1224	−0.1290	−0.1187	−0.1289	−0.1256	−0.1260	−0.1256	−0.1257
RMSE		0.0002	0.0186	0.1227	0.0170	0.0087	0.0012	0.0004	0.0001	0.0001

The "TRUE" value is based on the binomial tree model with $N = 10,000$ time steps. Columns 3–11 represent the binomial tree model with $N = 800$ time steps, the four-point method of Geske and Johnson (1984), the modified two-point Geske-Johnson method of Bunch and Johnson (1992), the four-point and six-point recursive schemes of Huang, Subrahmanyam, and Yu (1996), the lower and upper bound approximation of Broadie and Detemple (1996), the four-point and six-point randomization methods of Carr (1997), and the three-point multipiece exponential boundary method of this article, respectively. RMSE is the root of mean squared errors.

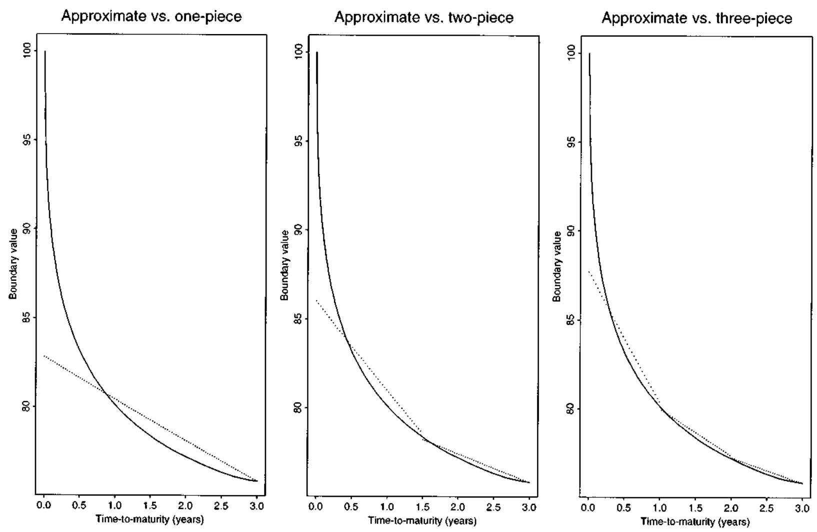

Figure 1
The approximate early exercise boundary versus multipiece exponential boundary for a put. Parameters used: $K = 100$, $T = 3$ (years), $\sigma = 0.2$, $r = 0.08$, $\delta = 0.04$. The solid line is the approximation to the true early exercise boundary using the finite-difference method. The dotted lines represent the one-piece, two-piece, and three-piece exponential boundary, respectively.

Even though we have only discussed standard American stock options, our approximation obviously applies to other standard American options such as futures options, quanto options, index options, and currency options. It is also worthwhile to point out that the present approximation applies equally well to other nonstandard American options when the early exercise premium can be represented as integrals involving the cumulative normal function. One particular example is the American barrier option.

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