Pricing Asian and Basket Options Via Taylor Expansion

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Abstract

Asian options belong to the so-called path-dependent derivatives. They are among the most difficult to price and hedge both analytically and numerically. Basket options are even harder to price and hedge because of the large number of state variables. Several approaches have been proposed in the literature, including Monte Carlo simulations, tree-based methods, partial differential equations, and analytical approximations among others. The last category is the most appealing because most of the other methods are very complex and slow. Our method belongs to the analytical approximation class. It is based on the observation that though the weighted average of lognormal variables is no longer lognormal, it can be approximated by a lognormal random variable if the first two moments match the true first two moments. To have a better approximation, we consider the Taylor expansion of the ratio of the characteristic function of the average to that of the approximating lognormal random variable around zero volatility. We include terms up to σ^6 in the expansion. The resulting option formulas are in closed form. We treat discrete Asian option as a special case of basket options. Formulas for continuous Asian options are obtained from their discrete counterpart. Numerical tests indicate that the formulas are very accurate. Comparisons with all other leading analytical approximations show that our method has performed the best overall in terms of accuracy for both short and long maturity options. Furthermore, unlike some other methods, our approximation treats basket (portfolio) and Asian options in a unified way. Lastly, in the appendix we point out a serious mathematical error of a popular method of pricing Asian options in the literature.

1 Introduction

Asian options belong to the so-called path-dependent derivatives. They are among the most difficult to price and hedge both analytically and numerically. Several approaches have been proposed in the literature. Since Boyle (1977) introduces it to the finance literature for option pricing, Monte Carlo simulation has been used by many authors. Kemna and Vorst (1990) use Monte Carlo simulation to price and hedge Asian options. For more recent development in simulation methods, see Broadie and Glasserman (1996) and Boyle, Broadie and Glasserman (1997). Although Monte Carlo simulation is a very flexible method for pricing path-dependent European options, it is very time-consuming.

By making a change of variables, Ingersoll (1987) and Wilmott, Dewynne, and Howison (1993) reduce the two-dimensional partial differential equation (PDE) satisfied by the price of a floating strike Asian option into a one-dimensional one. Rogers and Shi (1995) succeed doing the same for the fixed strike counterpart. This is a tremendous reduction of complexity in terms of computation. But the resulting PDE still requires numerical solutions. Based on the reduced one dimensional PDE, Zhang (2000) first derives an approximate formula for the Asian options. He then obtains the PDE for the difference between the true price and his approximate formula. He solves the PDE numerically. Zhang indicates that his analytical approximation coupled with his PDE achieves accuracy of the order of 10^{-5} for a wide range of parameters. Besides deriving a one dimensional PDE, Rogers and Shi (1995) also derive bounds of the Asian option prices. Recently, Thompson (1999) has improved upon their bounds. However, to obtain the bounds requires two-dimensional integrations. Another very accurate PDE based method is Hoogland and Neumann (2000). By working with only tradable assets, they obtain a PDE with no drift term. Solving such a PDE numerically is much easier because one does not have to deal with whether the PDE is of hyperbolic type or parabolic type.

Some other numerical methods include Hull and White (1993), Klassen (2000), Dewynne and Wilmott (1993) and Carverhill and Clewlow (1990). Hull and White (1993) and Klassen (2000) extend the binomial tree approach for pricing path-dependent options. They use a vector to hold the average rates at each node of the tree. Dewynne and Wilmott (1993) apply a similar idea to the PDE approach for pricing Asian options. Carverhill and Clewlow (1990) use the convolution method repeatedly to obtain the density function of the average rate in an Asian option. Even though these methods are simple to apply, they are all time-consuming. An exception is the Fast Fourier transformation method Carr and Madan (1999) when the characteristic function of the return is known analytically.

Our method belongs to the class of analytical approximations. Jarrow and Rudd (1982) seem to be the first to introduce the idea of Edgeworth expansion into the finance literature. Both Turnbull and Wakeman (1991) and Ritchken, Sankarasubramanian and Vijh (1993) use an Edgeworth series expansion to approximate the density function of the average rate. They obtain closed form formulas for the Asian options. Levy (1992) uses the lognormal density as a first-order approximation to the true density.[1] We demonstrate in section 3 that the lognormal approximation and the Edgeworth expansion method work fine for short maturities. However, for longer maturities these approximations cease to be reliable. Another analytical method is Geman and Yor (1993). They obtain a semi-analytical formula for the price of an Asian option using the Laplace transformation technique. Their analytical result is quite elegant, but it is a very difficult numerical problem to invert the Laplace transformation, see, for example, Fu, Madan and Wang (1999). For the latest development of the Laplace transformation approach, see Carr and Schröder (2001). Milevsky and Posner (1998a) approximate the density of the average rate with a reciprocal gamma distribution by matching the first two moments. Numerical evidences indicate that their approximation and that based on lognormal approximation have about the same level of accuracy, though it appears that the former is slightly more accurate. Posner and Milevsky (1998) and Milevsky and Posner (1998b) approximate the density function from the Johnson (1949) family by matching the first four moments. Our test indicates that their four-moment method outperforms all other existing approximations. However, it is not very accurate for long maturities Asian and basket options. Curran (1994) derives a pricing formula for Asian options by conditioning on the geometric mean. However, his formula also seems not to work well for basket options. Dufresne (2000) uses a Laguerre series to approximate Asian option prices, but his method fares poorly for short maturity options.

In light of the drawbacks of the other analytical methods, a reliable and simple analytical approximation is obviously highly desirable. We provide such an approximation in the next section. In section 3 we demonstrate that it is very accurate for a wide range of parameters for fixed strike European Asian and basket options. Our method develops a Taylor expansion around zero volatility. For more expansion methods, see Kunitomo and Takahashi (2001) and Reiner, Davydov and Kumanduri (2001).

The layout of the remainder of the article is as follows. The details of the analytical approximation are presented in section 2. Section 3 presents comparisons with all other leading analytical methods. It is demonstrated there that among the analytical methods, the present one is by far the most accurate. We conclude in section 4. In the appendix we point out a serious mathematical error in the Edgeworth expansion method in the literature.

2 Derivation of the Approximation

Even though the average of correlated lognormal random variables is no longer lognormally distributed, Levy (1992) demonstrates that lognormal distribution is a good approximation, especially for small maturities. To have a better approximation, we use Taylor expansion around zero volatilities to approximate the ratio of the characteristic function of the average to that of the approximating lognormal variable. This method is in spirit very similar to the perturbation method widely used in other disciplines, where an intractable problem is solved by approximating the solution around some small parameters. We consider the derivation in detail only for the basket options since Asian options can be treated as a special case in our method. Basket options are challenging because they can not be priced using the usual numerical methods like the partial differential equation (PDE) or the tree approach since the number of state variables may be too large. If the number of assets is small, the tree approach of Boyle, Evnine and Gibbs (1989) can be used.

2.1 Approximation for the Basket Options

We consider the following standard N-asset economy under the risk-neutral measure,[2]

$$S_i(t) = S_i e^{(g_i - \sigma_i^2/2)t + \sigma_i w_i(t)}, \qquad i = 1, 2, \cdots, N, \tag{1}$$

where $g_i = r - \delta_i$, r is the riskless interest rate, δ_i the dividend yield, σ_i the volatility, $w_i(t)$ a standard Wiener process. Let ρ_{ij} denote the correlation coefficients between $w_i(t)$ and $w_j(t)$.

At first glance it may seem that a method of Taylor expansion around zero volatilities does not apply because the volatility is different for each stock. We can overcome this difficulty by considering a fictitious market where all the individual volatilities are scaled by the same parameter z,

$$S_i(z, t) = S_i e^{(g_i - z^2\sigma_i^2/2)t + z\sigma_i w_i(t)}, \qquad i = 1, 2, \cdots, N. \tag{2}$$

Note that when $z = 1$, we recover the original processes.

Define

$$A(z) = \sum_{i=1}^{N} \chi_i S_i(z, T) = \sum_{i=1}^{N} \chi_i S_i e^{(g_i - z^2\sigma_i^2/2)T + z\sigma_i w_i(T)}, \tag{3}$$

where χ_i is the weight on stock i. The terminal payoff of a basket option is then given by (for a basket call)

$$BC(T) = (A(1) - K)^+, \tag{4}$$

where K is the strike price.

For simplicity, define $\bar{S}_i = \chi_i S_i e^{g_i T}$ and $\bar{\rho}_{ij} = \rho_{ij}\sigma_i\sigma_j T$. The first two moments of $A(z)$ are easily shown to be

$$U_1 = \sum_{i=1}^{N} \bar{S}_i = A(0), \tag{5}$$

$$U_2(z^2) = \sum_{ij=1}^{N} \bar{S}_i \bar{S}_j e^{z^2 \bar{\rho}_{ij}}. \tag{6}$$

Let $Y(z)$ be a normal random variable with mean $m(z^2)$ and variance $v(z^2)$. Matching the first two moments of $e^{Y(z)}$ with those of $A(z)$ we have

$$m(z^2) = 2\log U_1 - 0.5\log U_2(z^2), \tag{7}$$

$$v(z^2) = \log U_2(z^2) - 2\log U_1. \tag{8}$$

Let $X(z) = \log(A(z))$. We will try to find the density function of $X(z)$. To this end we consider its characteristic function,

$$E[e^{i\phi X(z)}] = E[e^{i\phi Y(z)}]\frac{E[e^{i\phi X(z)}]}{E[e^{i\phi Y(z)}]} = E[e^{i\phi Y(z)}]f(z), \tag{9}$$

where

$$E[e^{i\phi Y(z)}] = e^{i\phi m(z^2) - \phi^2 v(z^2)/2}$$

is the characteristic function of the normal random variable and

$$f(z) = \frac{E[e^{i\phi X(z)}]}{E[e^{i\phi Y(z)}]} = E[e^{i\phi X(z)}]e^{-i\phi m(z^2) + \phi^2 v(z^2)/2}$$

is the ratio of the characteristic function of $X(z)$ to that of $Y(z)$.

We expand $f(z)$ around $z = 0$ up to z^6.[3] First, we expand $e^{-i\phi m(z^2)+\phi^2 v(z^2)/2}$. Note that $v'(z^2) = -2m'(z^2)$.[4] Therefore we have

$$e^{-i\phi m(z^2)+\phi^2 v(z^2)/2} \approx e^{-i\phi m(0)+\phi^2 v(0)/2-(i\phi+\phi^2)m'(0)z^2-(i\phi+\phi^2)m''(0)z^4/2-(i\phi+\phi^2)m^{(3)}(0)z^6/6} \approx$$
$$e^{-i\phi m(0)+\phi^2 v(0)/2}(1-(i\phi+\phi^2)a_1+((i\phi+\phi^2)^2a_1^2-(i\phi+\phi^2)a_2)/2+$$
$$(3(i\phi+\phi^2)^2a_1a_2-(i\phi+\phi^2)a_3-(i\phi+\phi^2)^3a_1^3)/6), \tag{10}$$

where

$$\begin{aligned} a_1(z) &= z^2 m'(0) = -\frac{z^2 U_2'(0)}{2U_2(0)}, \\ a_2(z) &= z^4 m''(0) = 2a_1^2 - \frac{z^4 U_2''(0)}{2U_2(0)}, \\ a_3(z) &= z^6 m^{(3)}(0) = 6a_1a_2 - 4a_1^3 - \frac{z^6 U_2^{(3)}(0)}{2U_2(0)}, \end{aligned}$$

and

$$\begin{aligned} U_2(0) &= \sum_{ij=1}^{N} \bar{S}_i \bar{S}_j, \\ U_2'(0) &= \sum_{ij=1}^{N} \bar{S}_i \bar{S}_j (\bar{\rho}_{ij}), \\ U_2''(0) &= \sum_{ij=1}^{N} \bar{S}_i \bar{S}_j (\bar{\rho}_{ij})^2, \\ U_2^{(3)}(0) &= \sum_{ij=1}^{N} \bar{S}_i \bar{S}_j (\bar{\rho}_{ij})^3. \end{aligned}$$

The derivatives of $U_2(z^2)$ are with respect to z^2 because $U_2(z^2)$ is a function of z^2.

Now we approximate $g(z) = E[e^{i\phi X(z)}]$. Differentiating $g(z)$ twice yields

$$g^{(2)}(z) = E\left[e^{i\phi X(z)}(-(i\phi+\phi^2)X'^2(z) + i\phi\frac{A''(z)}{A(z)})\right]. \tag{11}$$

It can be easily checked that

$$E[\frac{A''(0)}{A(0)}] = 0,$$

and

$$z^2 E[X'^2(0)] = \frac{z^2}{A^2(0)} E[A'^2(0)] = -2a_1(z).$$

Therefore

$$\frac{z^2}{2} g^{(2)}(0) = e^{i\phi X(0)}(i\phi + \phi^2)a_1(z). \tag{12}$$

Differentiating $g(z)$ four times we have

$$\begin{aligned} g^{(4)}(z) &= E\Bigg[e^{i\phi X(z)}\Bigg(-(i\phi-3)(i\phi-2)(i\phi+\phi^2)X'^4(z) - 6(i\phi-2)(i\phi+\phi^2)\frac{A'^2(z)A''(z)}{A^3(z)} \\ &\quad -3(i\phi+\phi^2)\left(\frac{A''(z)}{A(z)}\right)^2 - 4(i\phi+\phi^2)\frac{A'(z)A^{(3)}(z)}{A^2(z)} + i\phi\frac{A^{(4)}(z)}{A(z)}\Bigg)\Bigg]. \end{aligned} \tag{13}$$

Straightforward calculation shows that

$$E[\frac{A'(0)A^{(3)}(0)}{A^2(0)}] = 0, \quad E[\frac{A^{(4)}(0)}{A(0)}] = 0.$$

Since $X'(0)$ is normally distributed with mean zero,

$$z^4 E[X'^4(0)] = z^4 3(E[X'^2(0)])^2 = 12a_1^2(z).$$

Therefore

$$\begin{aligned} \frac{z^4}{24} g^{(4)}(0) &= e^{i\phi X(0)}(-(i\phi-3)(i\phi-2)(i\phi+\phi^2)a_1^2(z)/2 - \\ &\quad (i\phi-2)(i\phi+\phi^2)b_1(z) - (i\phi+\phi^2)b_2(z)), \end{aligned} \tag{14}$$

where

$$\begin{aligned} b_1(z) &= \frac{z^4}{4A^3(0)} E[A'^2(0)A''(0)], \\ b_2(z) &= \frac{z^4}{8A^2(0)} E[A''^2(0)] = \frac{z^4 U_2''(0)}{4A^2(0)} = a_1^2(z) - \frac{1}{2}a_2(z), \end{aligned}$$

and

$$E[A'^2(0)A''(0)] = 2\sum_{ijk=1}^{N} \bar{S}_i \bar{S}_j \bar{S}_k \bar{\rho}_{ik} \bar{\rho}_{jk}.$$

The derivatives of $A(z)$ are with respect to z.

Differentiating $g(z)$ six times we have

$$
\begin{aligned}
g^{(6)}(z) &= E[e^{i\phi X(z)}(-(i\phi-5)(i\phi-4)(i\phi-3)(i\phi-2)(i\phi+\phi^2)X'^6(z) - \\
& 15(i\phi-4)(i\phi-3)(i\phi-2)(i\phi+\phi^2)\frac{A'^4(z)A''(z)}{A^5(z)} - \\
& (i\phi-3)(i\phi-2)(i\phi+\phi^2)(45(\frac{A'(z)}{A(z)})^2(\frac{A''(z)}{A(z)})^2 + 20\frac{A'^3(z)A^{(3)}(z)}{A^4(z)}) - \\
& (i\phi-2)(i\phi+\phi^2)(15(\frac{A'(z)}{A(z)})^2\frac{A^{(4)}(z)}{A(z)} + 60\frac{A'(z)A''(z)A^{(3)}(z)}{A^3(z)} + 15(\frac{A''(z)}{A(z)})^3) - \\
& (i\phi+\phi^2)(6\frac{A'(z)A^{(5)}(z)}{A^2(z)} + 10(\frac{A^{(3)}(z)}{A(z)})^2 + 15\frac{A''(z)A^{(4)}(z)}{A^2(z)}) + i\phi\frac{A^{(6)}(z)}{A(z)})]. \quad (15)
\end{aligned}
$$

It is easy to check that

$$
E[\frac{A'^2(0)A^{(4)}(0)}{{}^3A(0)}] = 0, \quad E[\frac{A'(0)A^{(5)}(0)}{A^2(0)}] = 0, \quad E[\frac{A''(0)A^{(4)}(0)}{A^2(0)}] = 0, \quad E[\frac{A^{(6)}(0)}{A(0)}] = 0,
$$

and

$$
z^6E[X'^6(0)] = z^6 15(E[X'^2(0)])^3 = -120a_1^3(z).
$$

Therefore

$$
\begin{aligned}
\frac{z^6}{720}g^{(6)}(0) &= e^{i\phi X(0)}(-(i\phi-5)(i\phi-4)(i\phi-3)(i\phi-2)(i\phi+\phi^2)(-\frac{a_1^3(z)}{6}) \\
& -(i\phi-4)(i\phi-3)(i\phi-2)(i\phi+\phi^2)c_1(z) - (i\phi-3)(i\phi-2)(i\phi+\phi^2)c_2(z) \\
& -(i\phi-2)(i\phi+\phi^2)c_3(z) - (i\phi+\phi^2)c_4(z)), \quad (16)
\end{aligned}
$$

where

$$
\begin{aligned}
c_1(z) &= \frac{z^6}{48A^5(0)}E[A'^4(0)A''(0)] = -a_1(z)b_1(z), \\
c_2(z) &= \frac{z^6}{144A^4(0)}(9E[A'^2(0)A''^2(0)] + 4E[A'^3(0)A^{(3)}(0)]), \\
c_3(z) &= \frac{z^6}{48A^3(0)}(4E[A'(0)A''(0)A^{(3)}(0)] + E[A''^3(0)]), \\
c_4(z) &= \frac{z^6}{72A^2(0)}E[(A^{(3)}(0))^2] = \frac{z^6U_2^{(3)}(0)}{12U_2(0)} = a_1(z)a_2(z) - \frac{2}{3}a_1^3(z) - \frac{1}{6}a_3(z),
\end{aligned}
$$

and

$$E[A^{'2}(0)A^{''2}(0)] = 8\sum_{ijkl=1}^{N} \bar{S}_i\bar{S}_j\bar{S}_k\bar{S}_l\bar{\rho}_{il}\bar{\rho}_{jk}\bar{\rho}_{kl} + 2U_2'(0)U_2''(0),$$
$$E[A^{'3}(0)A^{(3)}(0)] = 6\sum_{ijkl=1}^{N} \bar{S}_i\bar{S}_j\bar{S}_k\bar{S}_l\bar{\rho}_{il}\bar{\rho}_{jl}\bar{\rho}_{kl},$$
$$E[A'(0)A''(0)A^{(3)}(0)] = 6\sum_{ijk=1}^{N} \bar{S}_i\bar{S}_j\bar{S}_k\bar{\rho}_{ik}\bar{\rho}_{jk}^2,$$
$$E[A''^3(0)] = 8\sum_{ijk=1}^{N} \bar{S}_i\bar{S}_j\bar{S}_k\bar{\rho}_{ij}\bar{\rho}_{ik}\bar{\rho}_{jk}.$$

Finally, we have[5]

$$g(z) \approx g(0) + \frac{z^2}{2}g''(0) + \frac{z^4}{24}g^{(4)}(0) + \frac{z^6}{720}g^{(6)}(0), \tag{17}$$

where $g(0) = e^{i\phi X(0)}$. If we multiply (10) and (17), we have (up to terms with z^6) the ratio of the characteristic function of $X(z)$ to that of $Y(z)$ given by

$$f(z) \approx 1 - i\phi d_1(z) - \phi^2 d_2(z) + i\phi^3 d_3(z) + \phi^4 d_4(z), \tag{18}$$

where

$$\begin{aligned} d_1(z) &= \frac{1}{2}(6a_1^2(z) + a_2(z) - 4b_1(z) + 2b_2(z)) - \frac{1}{6}(120a_1^3(z) - a_3(z) + \\ &\quad 6(24c_1(z) - 6c_2(z) + 2c_3(z) - c_4(z))), & (19)\\ d_2(z) &= \frac{1}{2}(10a_1^2(z) + a_2(z) - 6b_1(z) + 2b_2(z)) - (128a_1^3(z)/3 - a_3(z)/6 + \\ &\quad 2a_1(z)b_1(z) - a_1(z)b_2(z) + 50c_1(z) - 11c_2(z) + 3c_3(z) - c_4(z)), & (20)\\ d_3(z) &= (2a_1^2(z) - b_1(z)) - \frac{1}{3}(88a_1^3(z) + 3a_1(z)(5b_1(z) - 2b_2(z)) + \\ &\quad 3(35c_1(z) - 6c_2(z) + c_3(z))), & (21)\\ d_4(z) &= (-20a_1^3(z)/3 + a_1(z)(-4b_1(z) + b_2(z)) - 10c_1(z) + c_2(z)). & (22) \end{aligned}$$

Note that $e^{i\phi X(0) - i\phi m(0) + \phi^2 v(0)/2} = 1$ has been used.

Finally $E[e^{i\phi X(1)}]$ is approximated by[6]

$$E[e^{i\phi X(1)}] \approx e^{i\phi m(1)-\phi^2 v(1)/2}(1 - i\phi d_1(1) - \phi^2 d_2(1) + i\phi^3 d_3(1) + \phi^4 d_4(1)), \tag{23}$$

and the density function of $X(1)$ by[7]

$$\begin{aligned} h(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\phi x} e^{i\phi m(1)-\phi^2 v(1)/2}(1 - i\phi d_1(1) - \phi^2 d_2(1) + i\phi^3 d_3(1) + \phi^4 d_4(1))d\phi \\ &= p(x) + (d_1(1)\frac{d}{dx} + d_2(1)\frac{d^2}{dx^2} + d_3(1)\frac{d^3}{dx^3} + d_4(1)\frac{d^4}{dx^4})p(x), \end{aligned} \tag{24}$$

where

$$p(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\phi x + i\phi m(1) - \phi^2 v(1)/2} d\phi = \frac{1}{\sqrt{2\pi v(1)}} e^{-\frac{(x-m(1))^2}{2v(1)}}$$

is the normal density with mean $m(1)$ and variance $v(1)$.

The price of a basket call is then given by

$$\begin{aligned} BC &= e^{-rT}E[e^{X(1)} - K]^+ = \left[U_1 e^{-rT}N(y_1) - Ke^{-rT}N(y_2)\right] + \\ & \left[e^{-rT}K\left(z_1 p(y) + z_2\frac{dp(y)}{dy} + z_3\frac{d^2p(y)}{dy^2}\right)\right], \end{aligned} \tag{25}$$

where

$$y = \log(K), \quad y_1 = \frac{m(1) - y}{\sqrt{v(1)}} + \sqrt{v(1)}, \quad y_2 = y_1 - \sqrt{v(1)},$$

and $z_1 = d_2(1) - d_3(1) + d_4(1)$, $z_2 = d_3(1) - d_4(1)$, $z_3 = d_4(1)$. Note that $d_1(1) - d_2(1) + d_3(1) - d_4(1) = 0$ has been used. The terms inside the first pair of square brackets give the price of the Levy (1992) approximation. The terms inside the second give the corrections. Note that (25) is in closed form and very simple.

The hedging ratio of a basket call is easily shown to be given by

$$\Delta_{BC} = \frac{\partial BC}{\partial S} = \frac{e^{-rT}U_1}{T}N(y_1) - \frac{e^{-rT}K}{S}\left(z_1\frac{dp(y)}{dx} + z_2\frac{d^2p(y)}{dx^2} + z_3\frac{d^3p(y)}{dx^3}\right). \tag{26}$$

The price of a basket put and its hedging ratio are given below by the call-put parity,

$$\begin{aligned} BP &= e^{-rT}K - e^{-rT}S\frac{U_1}{T} + BC, \\ \Delta_{BP} &= \frac{\partial BP}{\partial S} = \Delta_{BC} - e^{-rT}\frac{U_1}{T}. \end{aligned}$$

2.2 The Efficiency of the Approximation

Before we consider the Asian options in the next subsection, we comment on the efficiency of the proposed method. It appears that to obtain $c_2(1)$, it involves $O(N^4)$ calculations. If N is small, it poses no computational difficulty. However, if N is over 100, like for the S&P 500 index, N^4 will be huge. Fortunately, the method is really an N^3 algorithm.

To reduce the dimensionality of the problem, we note the following. If we define

$$\bar{A}_k = \sum_{i=1}^{N} \bar{S}_i \bar{\rho}_{ik}, \tag{27}$$

then

$$E[A'^2(0)A''(0)] = 2\sum_{k=1}^{N} \bar{S}_k \bar{A}_k^2. \tag{28}$$

Therefore to obtain $b_1(1)$, only $O(N^2)$ calculations are performed, though it appears that $O(N^3)$ are needed. Similarly,

$$E[A^{'2}(0)A''^2(0)] = 8\sum_{kl=1}^{N} \bar{A}_k \bar{S}_k \bar{\rho}_{kl} \bar{S}_l \bar{A}_l + 2U_2'(0)U_2''(0), \tag{29}$$

$$E[A^{'3}(0)A^{(3)}(0)] = 6\sum_{l=1}^{N} \bar{S}_l \bar{A}_l^3. \tag{30}$$

Therefore only $O(N^2)$ calculations are involved for $c_2(1)$. $E[A'(0)A''(0)A^{(3)}(0)]$ can be simplified as follows,

$$E[A'(0)A''(0)A^{(3)}(0)] = 6\sum_{jk=1}^{N} \bar{S}_j \bar{\rho}_{jk}^2 \bar{S}_k \bar{A}_k, \tag{31}$$

involving only $O(N^2)$ computations. If we define $\rho^*_{ij} = \sqrt{\bar{S}_i}\bar{\rho}_{ij}\sqrt{\bar{S}_j}$, then we can rewrite $E[A''^3(0)]$ as

$$E[A''^3(0)] = 8\sum_{ijk=1}^{N} \rho^*_{ij}\rho^*_{jk}\rho^*_{ki}. \tag{32}$$

Since ρ^*_{ij} is symmetric, the summations can be reduced to $i \leq j \leq k$, resulting in $O(\frac{N^3}{6})$ calculations. Therefore to obtain all the coefficients, it requires $O(\frac{N^3}{6})$ calculations, which is quite manageable even for an index like S&P 500. It is worthwhile to notice that the resulting formulas are very simple and easy to implement.

2.3 Approximation for the Asian Options

Define

$$A = \sum_{i=1}^{N} \chi_i S e^{(g-\sigma^2/2)t_i+\sigma w(t_i)}, \tag{33}$$

where χ_i is the weight of the stock price at time t_i, $g = r - \delta$, δ the dividend yield, σ the volatility, $t_1 = 0$ and $t_N = T$. If we define the new $\bar{S}_i$ and $\bar{\rho}_{ij}$ by $\bar{S}_i = \chi_i S e^{gt_i}$ and $\bar{\rho}_{ij} = \sigma^2 \min(t_i, t_j)$, respectively, the formulas for the basket options apply directly for the discrete Asian options. In cases where the weighting is the same for each stock price ($\chi_i = 1/N$) and the time interval between averaging points is the same ($t_{i+1} - t_i = T/(N-1) = \Delta$), closed form formulas can be obtained. The formulas for continuous Asian options are obatined by taking proper limits. For easy reference, they are provided in appendix A.

3 Performance Evaluation of the Approximation

In this section we evaluate the performance of the approximate formulas just derived. Since there is no closed form formula for Asian options, we need a reliable numerical method to give us the benchmark values. As discussed in the introduction, Zhang (2000) seems to give the most accurate prices among a number of numerical methods for continuous Asian options. We use Zhang (2000) to yield benchmark values for our comparisons. For the long maturity Asian options considered in table 2 and table 4, we also report the results from the tradable scheme method (TS) of Hoogland and Neumann (2000).[8] Though a PDE approach can be applied to the discrete Asian options, see Wilmott (1998) for more details, it is out of the question for the basket options if more than a few assets are included. For discrete Asian and basket options, we use the Monte Carlo simulation to generate the benchmark values. To reduce the standard deviations, we have adopted the antithetic variable technique and the control variate technique. For the latter we use the geometric mean option in Curran (1994) as our control variate. We use the root of mean squared error (RMSE) to measure the overall accuracy for a whole set of options and maximum absolute error (MAE) to gauge the worst possible case.

For continuously averaging Asian options, we consider the lognormal approximation (LN) of Levy (1992), the Edgeworth expansion method (EW) of Turnbull and Wakeman (1991) and Ritchken, Sankarasubramanian and Vijh (1993), the reciprocal gamma approximation (RG) of Milevsky and Posner (1998a), the four moment method (FM) of Posner and Milevsky (1998) and the method of this paper (TE6) for comparisons. For the discretely averaged Asian and basket options, we also consider the geometric conditioning approximation (GC) of Curran (1994). Since the Edgeworth expansion method does not appear to do better than the lognormal approximation, we only consider the latter for discrete Asian and basket options.

3.1 Continuously Averaging Asian Options

Table 1 considers continuous Asian options with moderate maturity. From the table it is clear that the present method is extremely accurate. For example, its RMSE is only 0.43 cents, while it is 9 cents for LN, 8.8 cents for EW, and 7.8 cents for RG. FM is also very accurate. Its RMSE is one third of 1 cent. The MAE of the present method is just slightly above 1 cent, but all the other methods except FM have large MAE's. FM's MAE is under one cent. For the options in table 1, the RMSE and MAE of the present method is comparable with those of FM and about 20 times or more smaller than those of the other methods. The performance of FM is really outstanding for these continuously averaging Asian options with moderate maturity. The following tests indicate that the performance of FM deteriorates for large volatility and long maturity options and also for basket options.

To test for the performance of these methods for longer maturities, we consider options with three years to maturity in Table 2. The improvement of the present method over the others is even more dramatic. The RMSE and MAE of it are more than 4 times smaller than those of FM and about 40 times or more smaller than those of the other methods. In fact, except FM, the other methods cease to give reliable values at all. The RMSE's of LN and RG are about 40 cents and the MAE's are about 80 cents. The most troubling is EW. Its RMSE is more than one dollar and its MAE is more than four dollars.

An important consideration for any method is how accurately it generates the hedging ratios. Table 3 reports the hedging ratios for the options considered in table 2. Considering that the present method gives very accurate prices, it is not surprising that it gives extremely accurate hedging ratios. FM yields even better results even though it performs worse than TE6 for the prices. For apprpximate methods, this can happen. For example, in Ju (1998), LUBA has about the same pricing errors as EXP3, but its hedging ratio errors are ten times larger. Note that EW gives completely wrong hedging ratios for large volatility options. For example, while the correct value for the third to last option is 0.62972, EW gives 0.06413, only about one tenth of the correct value. It appears that EW is completely unreliable for large maturity and volatility options. In the appendix we show that a serious mathematical mistake is made in using the lognormal density function in the Edgeworth expansion.[9] Therefore, it should not be expected to work well for moderate and long maturities and volatilities options.

3.2 Discretely Averaging Asian Options

So far we have only considered continuously averaging Asian options, though in practice they are all discretely averaged. We consider weekly averaging Asian options in table 4. We have run 0.1 and 4 million simulations for the first 9 (smaller volatilities) and second 9 (larger volatilities) options, respectively.[10] Note that TE6 performs about as well as it does for the continuous counterparts in table 2. Again LN and RG are not reliable for these long maturity options. GC has performed very well for this set of options. It seems that the geometric mean and the arithmetic mean are very highly correlated. This has also been indicated by the effectiveness of the geometric mean in reducing the standard errors in Monte Carlo simulations.

3.3 Basket (Portfolio) Options

We now consider basket options in tables 5 and 6. Each basket consists of 5 stocks, each with an initial price of 100. The weights are 0.05, 0.15, 0.2, 0.25, 0.35, respectively.[11] One million and four million simulations are run for the options in table 5 (shorter maturity) and table 6 (longer maturity), respectively. To reduce the variances, we also use the antithetic variable technique and use the Curran formula for basket options as our control variable. For the moderate maturity basket options in table 5, TE6 is extremely accurate. All other methods except FM have large pricing errors. To evaluate the performances for longer maturities, in table 6 we consider options with three years to maturity. Again TE6 performs very well. All the other methods have large pricing errors. The deterioration of the performances of GC and FM are noteworthy because they appear to have worked well for the Asian options when the maturity and volatility are not too large.

4 Concluding Remarks

We have proposed an extremely accurate analytical approximation for pricing Asian and basket options. The numerical tests in section 3 indicate that the approximate formulas can be used in most cases to achieve penny accuracy. Our method is also very easy and straightforward to implement. Lastly, we have pointed out a serious error in the literature in using the lognormal density in the Edgeworth expansion series.

One may think that the proposed method should work for the floating strike options too because the payoff function is a weighted sum of the stock prices at different time points. However, this is not so. For fixed strike Asian options, the average $A(T)$ is always positive. For floating strike ones, the random variable $A(T) - S(T)$ can be negative. Therefore the density of $A(T) - S(T)$ can not be approximated the same way as that of $A(T)$.

One potential application of the techniques developed is for expected utility calculations. In a lot of cases the expected utility is very difficult to obtain if the individual assets in a portfolio are assumed to follow lognormal processes because high dimension integrations are involved. To avoid this difficulty, the literature has usually assumed exponential utility functions and normal distributions for the asset values. For many assets, lognormal processes are obviously better approximations. Section 3 indicates that the approximate density function developed in section 2 for the basket is very accurate for most parameters encountered in practice. Therefore in cases where the portfolio weights are all positive (for example, with short sale constraints), the approximate density can be used to obtain the expected utility either in closed form or by a one dimensional integral.

Another potential application is for portfolio risk management. Normally it is very hard to assess the probability for a portfolio to reach a certain level some time in the future because so many random variables are involved. However, section 3 indicates that the future distribution of a portfolio can be accurately approximated by a simple one variable density function. Therefore the risk characteristics of a portfolio can be easily evaluated and the portfolio weights adjusted according to the desired risk characteristics.

A potential limitation of the paper is the assumption of log-normality of the underlying process. However, we point out that log-normal process is the only one well understood by traders in the industry and still widely used by them. Even in situationss where one models the volatility smile explicitly, the formulas developed can be used as a control variate in simulations by fixing the volatility at its average value.

Appendix A

This appendix provides the formulas for regular discrete Asian (equal weight and equal time interval) and continuous Asian options.[12]

$$U_1 = \frac{S}{N}\left(\frac{e^{gN\Delta}-1}{e^{g\Delta}-1}\right), \tag{34}$$

$$U_2 = \left(\frac{S}{N}\right)^2\left(\frac{e^{(2g+\sigma^2)Ndt}-2e^{gN\Delta}+1}{(e^{g\Delta}-1)(e^{(g+\sigma^2)\Delta}-1)} - \frac{(e^{(2g+\sigma^2)N\Delta}-1)e^{g\Delta}(e^{\sigma^2\Delta}-1)}{(e^{(2g+\sigma^2)\Delta}-1)(e^{g\Delta}-1)(e^{(g+\sigma^2)\Delta}-1)}\right), \tag{35}$$

and

$$\begin{aligned} z_1 = {} & \Big[(-\frac{1}{45}+\frac{1}{36N^2}-\frac{1}{180N^4})+(-\frac{1}{180}-\frac{1}{72N^2}+\frac{7}{360N^4})(Ng\Delta)+(\frac{11}{15120}-\frac{1}{720N^2}- \\ & \frac{1}{120N^4}+\frac{17}{1890N^6})(Ng\Delta)^2+(\frac{1}{2520}+\frac{1}{2160N^2}+\frac{7}{2160N^4}-\frac{31}{7560N^6})(Ng\Delta)^3+ \\ & (-\frac{1}{113400}+\frac{1}{12096N^2}+\frac{1}{7200N^4}+\frac{2}{567N^6}-\frac{377}{100800N^8})(Ng\Delta)^4\Big](N\sigma^2\Delta)^2+ \\ & \Big[(-\frac{1}{11340}-\frac{1}{1080N^2}-\frac{1}{135N^4}+\frac{191}{22680N^6})+(\frac{13}{30240}+\frac{17}{1440N^4}-\frac{37}{3024N^6}) \\ & (Ng\Delta)+(\frac{17}{226800}+\frac{13}{90720N^2}-\frac{13}{21600N^4}+\frac{2257}{90720N^6}-\frac{11111}{453600N^8})(Ng\Delta)^2+ \\ & (-\frac{23}{453600}+\frac{7}{43200N^4}-\frac{157}{18144N^6}+\frac{41}{4800N^8})(Ng\Delta)^3+(-\frac{59}{5987520}-\frac{23}{1814400N^2}+ \\ & \frac{23}{1088640N^4}+\frac{1481}{680400N^6}-\frac{1823}{72576N^8}+\frac{1373731}{59875200N^{10}})(Ng\Delta)^4\Big](N\sigma^2\Delta)^3, \end{aligned} \tag{36}$$

$$\begin{aligned} z_2 = {} & \Big[(-\frac{1}{90}+\frac{1}{72N^2}-\frac{1}{360N^4})+(-\frac{1}{360}-\frac{1}{144N^2}+\frac{7}{720N^4})(Ng\Delta)+(\frac{11}{30240}- \\ & \frac{1}{1440N^2}-\frac{1}{240N^4}+\frac{17}{3780N^6})(Ng\Delta)^2+(\frac{1}{5040}+\frac{1}{4320N^2}+\frac{7}{4320N^4}-\frac{31}{15120N^6}) \\ & (Ng\Delta)^3+(-\frac{1}{226800}+\frac{1}{24192N^2}+\frac{1}{1440N^4}+\frac{1}{567N^6}-\frac{377}{201600N^8})(Ng\Delta)^4\Big] \\ & (N\sigma^2\Delta)^2+\Big[(\frac{31}{22680}+\frac{7}{2160N^2}-\frac{11}{1080N^4}+\frac{253}{45360N^6})+(\frac{11}{60480}+\frac{1}{720N^2}+ \\ & \frac{13}{960N^4}-\frac{457}{30240N^6})(Ng\Delta)+(-\frac{37}{151200}-\frac{31}{181440N^2}-\frac{37}{14400N^4}+\frac{1307}{60480N^6}- \\ & \frac{16897}{907200N^8})(Ng\Delta)^2+(-\frac{19}{302400}-\frac{1}{6048N^2}-\frac{17}{86400N^4}-\frac{137}{12096N^6}+ \\ & \frac{2369}{201600N^8})(Ng\Delta)^3+(\frac{953}{59875200}-\frac{1}{518400N^2}+\frac{1013}{10886400N^4}+\frac{11051}{5443200N^6}- \\ & \frac{10579}{518400N^8}+\frac{2187809}{119750400N^{10}})(Ng\Delta)^4\Big](N\sigma^2\Delta)^3, \end{aligned} \tag{37}$$

$$
\begin{aligned}
z_3 &= \Big[(\frac{2}{2835} + \frac{1}{540N^2} - \frac{7}{2160N^4} + \frac{31}{45360N^6}) + (-\frac{1}{60480} + \frac{1}{1440N^2} + \frac{11}{2880N^4} - \\
&\quad \frac{17}{3780N^6})(Ng\Delta) + (-\frac{2}{14175} - \frac{11}{90720N^2} - \frac{49}{43200N^4} + \frac{13}{2835N^6} - \frac{2893}{907200N^8}) \\
&\quad (Ng\Delta)^2 + (-\frac{17}{907200} - \frac{1}{12096N^2} - \frac{1}{7200N^4} - \frac{127}{36288N^6} + \frac{377}{100800N^8})(Ng\Delta)^3 + \\
&\quad (\frac{13}{1247400} + \frac{1}{453600N^2} + \frac{449}{10886400N^4} + \frac{1709}{3628800N^6} - \frac{14239}{3628800N^8} + \\
&\quad \frac{407039}{119750400N^{10}})(Ng\Delta)^4\Big](N\sigma^2\Delta)^3. \qquad (38)
\end{aligned}
$$

With U_1, U_2, z_1, z_2 and z_3, (25) is a very simple approximation to price (regular) discrete Asian options. Note that equations (36)-(38) are not the exact expressions for z_1, z_2, z_3. The exact expressions are long and complex. We opt to report the final expressions for z_1, z_2, z_3 in terms of their Taylor expansion around $g = 0$. The coefficients of the terms with $(gN\Delta)^4$ are all much smaller than those with $(gN\Delta)^0$, indicating that no accuracy is likely to be lost for any reasonable $gN\Delta \approx gT$.[13]

The formulas for the continuously averaging case can be easily obtained by taking $N \longrightarrow \infty$ and $N\Delta = T$. For easy reference we report them here.

$$
\begin{aligned}
U_1 &= \frac{1}{g}(e^{gT} - 1) = A(0), \qquad (39)\\
U_2 &= \frac{2}{g+\sigma^2}\left(\frac{e^{(2g+\sigma^2)T} - 1}{2g+\sigma^2} - \frac{e^{gT} - 1}{g}\right), \qquad (40)
\end{aligned}
$$

and

$$
\begin{aligned}
z_1 &= -\sigma^4T^2(\frac{1}{45} + \frac{x}{180} - \frac{11x^2}{15120} - \frac{x^3}{2520} + \frac{x^4}{113400}) - \\
&\quad \sigma^6T^3(\frac{1}{11340} - \frac{13x}{30240} - \frac{17x^2}{226800} + \frac{23x^3}{453600} + \frac{59x^4}{5987520}), \qquad (41)\\
z_2 &= = -\sigma^4T^2(\frac{1}{90} + \frac{x}{360} - \frac{11x^2}{30240} - \frac{x^3}{5040} + \frac{x^4}{226800}) + \\
&\quad \sigma^6T^3(\frac{31}{22680} + \frac{11x}{60480} - \frac{37x^2}{151200} - \frac{19x^3}{302400} + \frac{953x^4}{59875200}), \qquad (42)\\
z_3 &= \sigma^6T^3(\frac{2}{2835} - \frac{x}{60480} - \frac{2x^2}{14175} - \frac{17x^3}{907200} + \frac{13x^4}{1247400}), \qquad (43)
\end{aligned}
$$

where $x = gT$. Note that (25) and (39)-(43) represent an extremely simple approximation for pricing continuous Asian options.

Appendix B

The numerical examples considered in the text clearly show that the Edgeworth expansion method yields completely unreliable results. In this appendix we show that Edgeworth expansion does not apply when it is used to approximate the density of the arithmetic average of a lognormal process.

The Edgeworth expansion technique amounts to approximate the ratio of the characteristic function of the random variable under consideration (F) to that of the approximating one (B)(A.4 in Jarrow and Rudd 1982) as follows,

$$\frac{\phi(F,t)}{\phi(B,t)} = \frac{E[e^{itF}]}{E[e^{itB}]} = \sum_{j=0}^{\infty} E_j \frac{(it)^j}{j!}, \tag{44}$$

where E_j is the coefficient for the j_{th} term. The first few coefficients are given in (A.5) in Jarrow and Rudd (1982) in terms of the cumulants. The cumulants are given in terms of moments (see 3.39 in Kendall and Stuart 1977.).

The essential idea is to represent e^{itF} and e^{itB} by their Taylor series and carry out the expectations term by term. However, the procedure breaks down if B is a lognormal random variable. The reason is that the series $S_n = \sum_{j=0}^{n} (it)^j E[B^j]/j!$ diverges. To see this assume $\log(B)$ is normal with mean m and variance v. Therefore $E[B^j] = e^{jm+0.5j^2v}$. The ratio test for convergence fails because

$$\left| e^{(j+1)m+0.5(j+1)^2v} \frac{(it)^{j+1}}{(j+1)!} \Big/ e^{jm+0.5j^2v} \frac{(it)^j}{j!} \right| = \frac{t}{j+1} e^{m+(j+0.5)v}$$

approaches infinity for each $t \neq 0$, $v \neq 0$ as $j \longrightarrow \infty$. Therefore $E[e^{itB}]$ cannot be approximated by its corresponding Taylor expansion representation. Unless the moments of F are related to those of B (lognormal) in such a way that the series in (44) converges, the Edgeworth expansion diverges when the approximating random variable B is lognormal.

Footnotes

1. In the literature, some authors have called the lognormal approximation the Turnbull and Wakeman method, see for example, Fu, Madan and Wang (1999) and Zhang (2000), but their original article includes correction terms.

2. One may argue that it may be more appropriate to assume that the portfolio value follows a lognormal process. However, the options market indicates that the Black-Scholes implied volatility curves for individual stock options are flatter than those of the index options. Therefore, the standard Black-Scholes model (1973) seems to approximate the risk-neutral processes better for individual stocks than for indexes.

3. Since z is a scaling parameter for the volatilities, this is equivalent to up to σ_i^6.

4. The derivatives are with respect to z^2 because $m(z^2)$ and $v(z^2)$ are functions of z^2.

5. Note that $g(z)$ depends only on even orders of z. This is verified by direct computation. Intuitively, $g(z)$ must be an even function of z since the process $dS/S = \mu dt + z\sigma dw$ is statistically the same as $dS/S = \mu dt - z\sigma dw$.

6. Note that the above equation has the same structure as the Edgeworth expansion in the appendix that the approximate charateristic function is that of the approximating variable times a polynomial of ϕ. However, the coefficients are different. Unlike in the Edgeworth expansion, in our approximation the coefficients for the powers of ϕ change if more terms of powers of z are included. Another difference is that we have considered $\log(A(T))$ instead of $A(T)$.

7. If we had used the expansion of $E[e^{i\phi X(1)}]$ directly, we would have $E[e^{i\phi X(1)}] \approx e^{i\phi X(0)}(1 + i\phi e_1 + \phi^2 e_2 + i\phi^3 e_3 + \phi^4 e_4 + i\phi^5 e_5 + \phi^6 e_6)$, where the coefficients $e_1 - e_6$ are independent

of ϕ. Fourier inversion would yield an approximate density function which involves the Dirac delta function and its derivatives. Such an approximation is not a good one.

8. We thank Dimitri Neumann for providing us the values. We regard these values as accurate too. Therefore we do not report error measures for them.

9. We have become aware that the divergence problem with EW has been known to a number of practitioners and academics. For lack of reference, we provide a formal proof of the divergence in appendix B.

10. To reduce the simulation time, the same random numbers are used for all options. This explains the same standard errors for the options with the same volatility. This is equivalent to starting the simulations with the same seed for the random number generator and should not affect the accuarcy of each price.

11. These weights are chosen arbitrarily. Equivalently, with equal weighting (0.2), the prices are 5, 15, 20, 25, 35, respectively. Note that more assets could be considered, but the Monte Carlo simulations for the benchmark values would take too long.

12. Mathematica is used to obtain the following formulas for the regular discrete Asian options.

13. We have used the exact expressions in our numerical tests. Identical results are obtained.

References

Alziary, B., J. Decamps, P. Koehl, 1997. A P.D.E. approach to Asian Options: Analytical and Numerical Evidence. *Journal of Banking and Finance*, 21, 613-640.

Black, F., M. Scholes, 1973. The Pricing of Options and Corporate Liabilities. *Journal of Political Economy*, 81, 637-659.

Boyle, P., 1977. Options: A Monte Carlo Approach. *Journal of Financial Economics*, 4, 323-338.

Boyle, P., M. Broadie, P. Glasserman, 1997. Monte Carlo Methods for Security Pricing. *Journal of Economic Dynamics and Control*, 21, 1267-1321.

Boyle, P., J. Evnine, S. Gibbs, 1989. Numerical Evaluation of Multivariate Contingent Claims. *Review of Financial Studies*, 2, 241-250.

Broadie, M., P. Glasserman, 1996. Estimating Security Price Derivatives Using Simulations. *Management Science*, 42, 269-285.

Carr, P., D. Madan, 1999. Option Valuation using the Fast Fourier Transformation. *Journal of Computational Finance* 2, 61-73.

Carr, P., M. Schröder, 2001. On the Valuation of Arithmetic-Average Asian Options: the Geman-Yor Laplace Transformation Revisited. Banc of America Securities, Working Paper.

Carverhill, A., L. Clewlow, 1990. Flexible Convolution. *Risk*, 3, 25-29.

Curran, M., 1994. Valuing Asian and Portfolio Options by Conditioning on the Geometric Mean Price. *Management Science*, 40, 1705-1711.

Dewynne, J., P. Wilmott, 1993. Partial to the Exotic. *Risk*, 6, 38-46.

Dufresne, D., 2000. Laguerre Series for Asian and Other Options. Forthcoming in *Mathematical Finance*.

Fu, M., D. Madan, T. Wang, 1999. Pricing Continuous Asian Options: A Comparison of Monte Carlo and Laplace Transform Inversion Methods. *Journal of Computational Finance*, 2, 49-74.

Geman, H., M. Yor, 1993. Bessel Process, Asian Options, and Perpetuities. *Mathematical Finance*, 4, 349-375.

Hoogland,J. K., C. D. D. Neumann, 2000. Tradable schemes. Technical Report MAS-0024, CWI, Amsterdam, The Netherlands.

Hull, J., A. While, 1993. Efficient Procedures for Valuing European and American path-dependent Options. *Journal of Derivatives*, 1, 21-31.

Ingersoll, J., 1987. Theory of Financial Decision Making. Rowman & Littlefield publishers, Inc.

Jarrow, R., A. Rudd, 1982. Approximate Option Valuation for Arbitrary Stochastic Processes. *Journal of Financial Economics*, 10, 347-369.

Johnson, N., 1949. Systems of Frequency Curves Generated by methods of Translation. *Biometrika*, 36, 149-176.

Ju, N., 1998. Pricing an American Option by Approximating its Early Exercise Boundaries as a Multipiece Exponential Function. *Review of Financial Studies*, 11, 627-646.

Kemna, A., A. Vorst, 1990. A Pricing Method for Options Based on Average Asset Values. *Journal of Banking and Finance*, 14, 113-130.

Kendall, M., A. Stuart, 1977. The Advanced Theory of Statistics. Vol. 1, Macmillan Publishing Co, New York.

Klassen, T., 2000. Simple, Fast and Flexible Pricing of Asian Options. Forthcoming in *Journal of Computational Finance.*

Kunitomo, N., A. Takahashi, 2001. The Asymptotic Expansion Approach to Valuation of Interest Rate Contingent Claims. *Mathematical Finance* 11, 117-151.

Levy, E., 1992. Pricing European Average rate Currency Options. *Journal of International Money and Finance*, 11, 474-491.

Milevsky, M., S. Posner, 1998a. Asian Options, the Sum of Lognormals, and the Reciprocal Gamma Distribution. *journal of Financial and Quantitative Analysis*, 33, 409-422.

Milevsky, M., S. Posner, 1998b. A Closed-Form Approximation for Valuing Basket Options. *Journal of Derivatives*, 5, 54-61.

Posner, S., M. Milevsky, 1998. Valuing Exotic Options by Approximating the SPD with Higher Moments. *Journal of Financial Engineering*, 7, 109-125.

Reiner, E., D. Davydov, R. Kumanduri, 2001. A Rapidly Convergent Expansion Method for Asian and Basket Options. Working Paper, UBS Warburg.

Ritchken, P., L. Sankarasubramanian, A. Vijh, 1993. The Valuation of Path Dependent Contracts on the Average. *Management Science*, 1202-1213.

Rogers, L., Z. Shi, 1995. The value of an Asian Option. *Journal of Applied Probability*, 32, 1077-1088.

Thompson, G., 1999. Fast narrow bounds on the value of Asian options. Working Paper, Center for Financial Research, Judge Institute of Management Science, University of Cambridge.

Turnbull, S., L. Wakeman, 1991. A Quick Algorithm for Pricing European Average Options *Journal of Financial and Quantitative Analysis*, 26, 377-389.

Wilmott, P., J. Dewynne, S. Howison, 1993. Option Pricing: Mathematical Models and Computation. Oxford Finacial Press.

Zhang, J., 2000. Arithmetic Asian options with Continuous Sampling. Forthcoming in *Journal of Computational Finance.*

Table 1: Values of Continuous Averaging Calls ($S = 100$, $r = 0.09$, $\delta = 0$, $T = 1$)

(1)	(2)	(3)	(4)	(5)	(6)	(7)
(σ, K)	Exact	TE6	LN	EW	RG	FM
(0.05, 95)	8.80884	8.80884	8.80888	8.80884	8.80881	8.80884
(0.05, 100)	4.30823	4.30824	4.30972	4.30823	4.30720	4.30823
(0.05, 105)	0.95838	0.95837	0.95815	0.95838	0.95851	0.95838
(0.1, 95)	8.91185	8.91190	8.91721	8.91185	8.90820	8.91186
(0.1, 100)	4.91512	4.91513	4.92310	4.91459	4.90938	4.91512
(0.1, 105)	2.07006	2.06996	2.07045	2.07002	2.06952	2.07006
(0.2, 95)	9.99565	9.99594	10.03043	9.98596	9.97052	9.99552
(0.2, 100)	6.77735	6.77692	6.80355	6.77025	6.75716	6.77720
(0.2, 105)	4.29646	4.29561	4.30409	4.29618	4.28890	4.29641
(0.3, 95)	11.65590	11.65565	11.73288	11.60606	11.59733	11.65500
(0.3, 100)	8.82876	8.82686	8.88576	8.80190	8.78217	8.82792
(0.3, 105)	6.51779	6.51494	6.54628	6.51750	6.49026	6.51726
(0.4, 95)	13.51072	13.50887	13.64791	13.36950	13.40169	13.50764
(0.4, 100)	10.92378	10.91903	11.03113	10.85700	10.83223	10.92085
(0.4, 105)	8.72994	8.72337	8.79965	8.73326	8.66299	8.72764
(0.5, 95)	15.44269	15.43806	15.66486	15.13556	15.25983	15.43448
(0.5, 100)	13.02817	13.01899	13.21198	12.89936	12.86687	13.02013
(0.5, 105)	10.92963	10.91731	11.06752	10.95397	10.79735	10.92260
RMSE		0.00434	0.09110	0.08796	0.07789	0.00339
MAE		0.01232	0.22217	0.30713	0.18286	0.00821

The "Exact" value is obtained using the method in Zhang (2000). Columns 3-7 represent the Taylor expansion (TE6) approach of this article, the lognormal approximation (LN) of Levy (1992), the Edgeworth expansion approximation (EW) of Turnbull and Wakeman (1991) and Ritchken, Sankarasubramanian and Vijh (1993) the reciprocal gamma distribution method (RG) of Milevsky and Posner (1998a), and the four-moment approximation (FM) of Posner and Milevsky (1998), respectively. RMSE is the root of mean squared errors and MAE is the maximum absolute error.

Table 2: Values of Continuous Averaging Calls ($S = 100$, $r = 0.09$, $\delta = 0$, $T = 3$)

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
(σ, K)	Exact	TS	TE6	LN	EW	RG	FM
(0.05, 95)	15.11626	15.11627	15.11626	15.11630	15.11628	15.11624	15.11626
(0.05, 100)	11.30361	11.30361	11.30360	11.30422	11.30368	11.30318	11.30361
(0.05, 105)	7.55332	7.55332	7.55335	7.55670	7.55335	7.55075	7.55333
(0.1, 95)	15.21380	15.21380	15.21396	15.22546	15.21443	15.20538	15.21383
(0.1, 100)	11.63766	11.63766	11.63798	11.65759	11.63510	11.62237	11.63764
(0.1, 105)	8.39122	8.39122	8.39140	8.41475	8.38630	8.37232	8.39115
(0.2, 95)	16.63723	16.63721	16.63942	16.74023	16.52766	16.55504	16.63634
(0.2, 100)	13.76693	13.76693	13.76770	13.86951	13.66580	13.68133	13.76559
(0.2, 105)	11.21985	11.21987	11.21879	11.31054	11.14619	11.14037	11.21835
(0.3, 95)	19.02320	19.02316	19.02652	19.27910	18.36063	18.80529	19.01620
(0.3, 100)	16.58613	16.58613	16.58509	16.82823	16.10382	16.37079	16.57768
(0.3, 105)	14.39295	14.39298	14.38751	14.61010	14.09940	14.19042	14.38394
(0.4, 95)	21.74097	21.74092	21.74461	22.23180	19.63683	21.30176	21.71507
(0.4, 100)	19.58830	19.58825	19.58355	20.05569	18.26416	19.15285	19.55790
(0.4, 105)	17.62548	17.62544	17.61269	18.05875	16.98036	17.20469	17.59266
(0.5, 95)	24.57190	24.57187	24.57740	25.40607	20.28830	23.79644	24.50412
(0.5, 100)	22.63085	22.63078	22.62276	23.43633	20.82553	21.85456	22.55035
(0.5, 105)	20.84322	20.84318	20.82213	21.60941	21.00517	20.07802	20.75420
RMSE			0.00662	0.39303	1.26967	0.37305	0.03486
MAE			0.02108	0.83417	4.28360	0.77629	0.08902

The "Exact" value is obtained using the method in Zhang (2000). Columns 3-8 represent the tradable scheme (TS) method of Hoogland and Neumann (2000), Taylor expansion (TE6) approach of this article, the lognormal approximation (LN) of Levy (1992), the Edgeworth expansion approximation (EW) of Turnbull and Wakeman (1991) and Ritchken, Sankarasubramanian and Vijh (1993) the reciprocal gamma distribution method (RG) of Milevsky and Posner (1998a), and the four-moment approximation (FM) of Posner and Milevsky (1998), respectively. RMSE is the root of mean squared errors and MAE is the maximum absolute error.

Table 3: Hedging Ratios (Δ's) of Continuous Averaging Calls ($S = 100$, $r = 0.09$, $\delta = 0$, $T = 3$)

(1) (σ, K)	(2) Exact	(3) TE6	(4) LN	(5) EW	(6) RG	(7) FM
(0.05, 95)	0.87630	0.87630	0.87627	0.87629	0.87631	0.87630
(0.05, 100)	0.87357	0.87357	0.87329	0.87356	0.87377	0.87357
(0.05, 105)	0.84217	0.84216	0.84141	0.84222	0.84278	0.84217
(0.1, 95)	0.85158	0.85153	0.85009	0.85205	0.85276	0.85158
(0.1, 100)	0.80488	0.80487	0.80367	0.80552	0.80596	0.80489
(0.1, 105)	0.71793	0.71800	0.71831	0.71806	0.71780	0.71793
(0.2, 95)	0.74051	0.74074	0.74052	0.74009	0.74123	0.74062
(0.2, 100)	0.68025	0.68061	0.68255	0.67537	0.67912	0.68030
(0.2, 105)	0.61171	0.61209	0.61621	0.60386	0.60869	0.61170
(0.3, 95)	0.67654	0.67738	0.68054	0.63823	0.67496	0.67685
(0.3, 100)	0.62877	0.62966	0.63515	0.58619	0.62502	0.62888
(0.3, 105)	0.57924	0.58008	0.58764	0.53743	0.57359	0.57919
(0.4, 95)	0.64497	0.64661	0.65318	0.46790	0.64103	0.64579
(0.4, 100)	0.60712	0.60875	0.61765	0.44724	0.60087	0.60751
(0.4, 105)	0.56917	0.57066	0.58161	0.43199	0.56088	0.56916
(0.5, 95)	0.62972	0.63235	0.64233	0.06413	0.62335	0.63186
(0.5, 100)	0.59888	0.60145	0.61379	0.13813	0.59000	0.60016
(0.5, 105)	0.56841	0.57083	0.58527	0.20641	0.55731	0.56890
RMSE		0.00128	0.00801	0.20332	0.00488	0.00064
MAE		0.00263	0.01686	0.56559	0.01110	0.00214

The "Exact" value is obtained using the method in Zhang (2000). Columns 3-7 represent the Taylor expansion (TE6) approach of this article, the lognormal approximation (LN) of Levy (1992), the Edgeworth expansion approximation (EW) of Turnbull and Wakeman (1991) and Ritchken, Sankarasubramanian and Vijh (1993) the reciprocal gamma distribution method (RG) of Milevsky and Posner (1998a), and the four-moment approximation (FM) of Posner and Milevsky (1998), respectively. RMSE is the root of mean squared errors and MAE is the maximum absolute error.

Table 4: Values of Weekly Averaging Calls ($S = 100$, $r = 0.09$, $\delta = 0$, $T = 3$)

(1) (σ, K)	(2) MC (SD)	(3) TS	(4) TE6	(5) LN	(6) RG	(7) FM	(8) GC
(0.05, 95)	15.1199 (0.0002)	15.1197	15.1197	15.1197	15.11965	15.1197	15.1197
(0.05, 100)	11.3071 (0.0002)	11.3070	11.3069	11.3076	11.30654	11.3070	11.3069
(0.05, 105)	7.5563 (0.0002)	07.5561	7.5562	7.5596	7.55364	7.5561	7.5561
(0.1, 95)	15.2171 (0.0007)	15.2163	15.2165	15.2281	15.20814	15.2163	15.2163
(0.1, 100)	11.6399 (0.0007)	11.6390	11.6394	11.6593	11.62415	11.6390	11.6389
(0.1, 105)	8.3919 (0.0007)	8.3911	8.3913	8.4150	8.37269	8.3911	8.3907
(0.2, 95)	16.6366 (0.0026)	16.6342	16.6365	16.7388	16.55403	16.6333	16.6331
(0.2, 100)	13.7654 (0.0026)	13.7626	13.7634	13.8668	13.67896	13.7612	13.7611
(0.2, 105)	11.2174 (0.0026)	11.2146	11.2135	11.3066	11.13678	11.2130	11.2125
(0.3, 95)	19.0194 (0.0009)	19.0144	19.0179	19.2743	18.80143	19.0075	19.0098
(0.3, 100)	16.5812 (0.0009)	16.5766	16.5755	16.8224	16.36582	16.5681	16.5715
(0.3, 105)	14.3874 (0.0009)	14.3830	14.3774	14.6034	14.18451	14.3739	14.3771
(0.4, 95)	21.7331 (0.0018)	21.7269	21.7307	22.2251	21.29668	21.7010	21.7150
(0.4, 100)	19.5798 (0.0018)	19.5738	19.5690	20.0481	19.14690	19.5434	19.5615
(0.4, 105)	17.6164 (0.0018)	17.6109	17.5978	18.0505	17.19800	17.5780	17.5977
(0.5, 95)	24.5585 (0.0031)	24.5526	24.5583	25.3991	23.79172	24.4848	24.5286
(0.5, 100)	22.6167 (0.0031)	22.6115	22.6032	23.4287	21.84918	22.5308	22.5871
(0.5, 105)	20.8289 (0.0031)	20.8241	20.8023	21.6012	20.07208	20.7348	20.7990
RMSE			0.0092	0.3952	0.3696	0.0380	0.0150
MAE			0.0266	0.8406	0.7675	0.0941	0.0299

Columns 2-7 represent the Monte Carlo simulation (standard deviation), the tradable scheme (TS) method of Hoogland and Neumann 2000, the Taylor expansion (TE6) approach of this article, the lognormal approximation (LN) of Levy 1992, the reciprocal gamma distribution method (RG) of Milevsky and Posner 1998a, the four-moment approximation (FM) of Posner and Milevsky 1998, the geometric conditioning method (GC) of Curran 1994, respectively. RMSE is the root of mean squared errors and MAE is the maximum absolute error.

Table 5: Values of Basket Calls ($\delta = 0$, $T = 1$)

(1) (K, r, σ, ρ)	(2) MC (SD)	(3) TE6	(4) LN	(5) RG	(6) FM	(7) GC
(90 ,0.05 ,0.2 ,0.0)	14.6254 (0.0011)	14.6259	14.6372	14.6058	14.6259	14.6193
(100 ,0.10 ,0.2 ,0.0)	10.3070 (0.0011)	10.3087	10.3255	10.2803	10.3084	10.2956
(110 ,0.05 ,0.5 ,0.0)	8.4260 (0.005)	8.4268	8.5011	8.3729	8.3943	8.0436
(90 ,0.10 ,0.5 ,0.0)	21.2996 (0.0065)	21.3083	21.4717	21.1135	21.2968	21.0793
(100 ,0.05 ,0.2 ,0.5)	8.8929 (0.0004)	8.8933	8.8947	8.8042	8.8933	8.8903
(110 ,0.10 ,0.2 ,0.5)	6.5267 (0.0003)	6.5272	6.5280	6.4776	6.5272	6.5239
(90 ,0.05 ,0.5 ,0.5)	22.8694 (0.0029)	22.8738	22.8899	21.9817	22.8716	22.8213
(100 ,0.10 ,0.5 ,0.5)	20.2037 (0.0028)	20.2014	20.2165	19.3612	20.1989	20.1446
(110 ,0.05 ,0.2 ,0.0)	2.2074 (0.0007)	2.2071	2.2016	2.2186	2.2072	2.1855
(90 ,0.10 ,0.2 ,0.0)	18.6285 (0.0012)	18.6286	18.6342	18.6187	18.6288	18.6259
(100 ,0.05 ,0.5 ,0.0)	12.6438 (0.0054)	12.6480	12.7871	12.5118	12.6123	12.2990
(110 ,0.10 ,0.5 ,0.0)	10.5148 (0.0052)	10.5184	10.6303	10.4195	10.4826	10.1480
(90 ,0.05 ,0.2 ,0.5)	15.6479 (0.0005)	15.6477	15.6494	15.5330	15.6477	15.6458
(100 ,0.10 ,0.2 ,0.5)	11.9199 (0.0005)	11.9198	11.9215	11.8055	11.9197	11.9172
(110 ,0.05 ,0.5 ,0.5)	13.8766 (0.0028)	13.8818	13.8918	13.3080	13.8794	13.8179
(90 ,0.10 ,0.5 ,0.5)	25.3757 (0.0031)	25.3810	25.3975	24.4698	25.3791	25.3329
(100 ,0.05 ,0.2 ,0.0)	6.8143 (0.0009)	6.8154	6.8308	6.7919	6.8150	6.7964
(110 ,0.10 ,0.2 ,0.0)	4.2398 (0.0007)	4.2396	4.2466	4.2317	4.2395	4.2175
(90 ,0.05 ,0.5 ,0.0)	18.3388 (0.0062)	18.3360	18.5035	18.1457	18.3135	18.0636
(100 ,0.10 ,0.5 ,0.0)	15.2241 (0.0058)	15.2322	15.3912	15.0634	15.2006	14.9159
(110 ,0.05 ,0.2 ,0.5)	4.3969 (0.0004)	4.3967	4.3967	4.3981	4.3967	4.3934
(90 ,0.10 ,0.2 ,0.5)	19.2149 (0.0006)	19.2149	19.2163	19.1192	19.2149	19.2136
(100 ,0.05 ,0.5 ,0.5)	17.8991 (0.0028)	17.9159	17.9022	17.1347	17.8996	17.8422
(110 ,0.10 ,0.5 ,0.5)	15.9268 (0.0028)	15.9274	15.9395	15.2444	15.9248	15.8652
RMSE		0.0033	0.0629	0.3905	0.0131	0.1631
MAE		0.0087	0.1721	0.9059	0.0322	0.3824

Columns 2-7 represent the Monte Carlo simulation (standard deviation), the Taylor expansion (TE6) approach of this article, the lognormal approximation (LN) of Levy (1992), the reciprocal gamma distribution method (RG) of Milevsky and Posner (1998a), the four-moment approximation (FM) of Posner and Milevsky (1998), the geometric conditioning method (GC) of Curran (1994), respectively. RMSE is the root of mean squared errors and MAE is the maximum absolute error. Five stocks are included in each basket, each with an initial price of 100. The weights are 0.05, 0.15, 0.2, 0.25 and 0.35, respectively. The volatilities and correlations are assumed to be the same.

Table 6: Values of Basket Calls ($\delta = 0$, $T = 3$)

(1) (K, r, σ, ρ)	(2) MC (SD)	(3) TE6	(4) LN	(5) RG	(6) FM	(7) GC
(90, 0.05, 0.2, 0.0)	23.0121 (0.0017)	23.0148	23.0561	22.9501	23.0169	22.9777
(100, 0.10, 0.2, 0.0)	26.1671 (0.0017)	26.1706	26.2005	26.1232	26.1738	26.1463
(110, 0.05, 0.5, 0.0)	21.0184 (0.0098)	21.0437	21.8495	20.4446	20.5303	19.3543
(90, 0.10, 0.5, 0.0)	37.2287 (0.0107)	37.1973	37.9690	36.5845	37.3486	36.3446
(100, 0.05, 0.2, 0.5)	18.5798 (0.0007)	18.5812	18.5875	18.1930	18.5809	18.5653
(110, 0.10, 0.2, 0.5)	21.7596 (0.0008)	21.7600	21.7664	21.3672	21.7598	21.7465
(90, 0.05, 0.5, 0.5)	36.8280 (0.0057)	36.8255	36.9131	33.4933	36.8083	36.5786
(100, 0.10, 0.5, 0.5)	38.5906 (0.0057)	38.5874	38.6742	35.3043	38.5789	38.3534
(110, 0.05, 0.2, 0.0)	9.8016 (0.0013)	9.8013	9.8546	9.7366	9.7960	9.6887
(90, 0.10, 0.2, 0.0)	33.3711 (0.0018)	33.3707	33.3810	33.3551	33.3735	33.3639
(100, 0.05, 0.5, 0.0)	25.1610 (0.0100)	25.1394	26.0042	24.4746	24.7427	23.6466
(110, 0.10, 0.5, 0.0)	27.6233 (0.0101)	27.6190	28.4929	26.9366	27.3201	26.2559
(90, 0.05, 0.2, 0.5)	24.8104 (0.0008)	24.8111	24.8172	24.4386	24.8110	24.7999
(100, 0.10, 0.2, 0.5)	27.5462 (0.0008)	27.5463	27.5519	27.2072	27.5463	27.5370
(110, 0.05, 0.5, 0.5)	29.1034 (0.0059)	29.1026	29.1871	25.8326	29.0556	28.8042
(90, 0.10, 0.5, 0.5)	42.7673 (0.0058)	42.7625	42.8455	39.6780	42.7760	42.5596
(100, 0.05, 0.2, 0.0)	15.6780 (0.0016)	15.6802	15.7425	15.5903	15.6775	15.6027
(110, 0.10, 0.2, 0.0)	19.4368 (0.0016)	19.4357	19.4894	19.3541	19.4358	19.3805
(90, 0.05, 0.5, 0.0)	29.9998 (0.0103)	29.9817	30.8485	29.2973	29.7887	28.7457
(100, 0.10, 0.5, 0.0)	32.1145 (0.0104)	32.1032	32.9523	31.4284	32.0113	30.9819
(110, 0.05, 0.2, 0.5)	13.4902 (0.0006)	13.4905	13.4954	13.1811	13.4901	13.4713
(90, 0.10, 0.2, 0.5)	34.0088 (0.0008)	34.0101	34.0140	33.7796	34.0102	34.0047
(100, 0.05, 0.5, 0.5)	32.7054 (0.0058)	32.7176	32.8051	29.3599	32.6827	32.4421
(110, 0.10, 0.5, 0.5)	34.8364 (0.0057)	34.8388	34.9267	31.4782	34.8126	34.5777
RMSE		0.0108	0.4177	1.6834	0.1553	0.6764
MAE		0.0314	0.8696	3.3582	0.4881	1.6641

Columns 2-7 represent the Monte Carlo simulation (standard deviation), the Taylor expansion (TE6) approach of this article, the lognormal approximation (LN) of Levy (1992), the reciprocal gamma distribution method (RG) of Milevsky and Posner (1998a), the four-moment approximation (FM) of Posner and Milevsky (1998), the geometric conditioning method (GC) of Curran (1994), respectively. RMSE is the root of mean squared errors and MAE is the maximum absolute error. Five stocks are included in each basket, each with an initial price of 100. The weights are 0.05, 0.15, 0.2, 0.25 and 0.35, respectively. The volatilities and correlations are assumed to be the same.