Delegated Portfolio Management under Adverse Selection in a Continuous-Time Model

Nengjiu Ju* and Xuhu Wan†

October 1, 2011

ABSTRACT

This paper studies the optimal contracting problem between a representative fund investor and a constant absolute risk-aversion fund manager whose skill is unobservable by the investor. The optimal compensation contract and the manager's optimal fund investment policy are derived in closed form. Several interesting results are obtained. First, the optimal contract involves an incentive fee which is symmetric around a market-based benchmark, complementing existing result on the optimality of symmetric incentive fees when the investor and the manager have symmetric information and justifying the use of symmetric incentive fees even when there is asymmetric information. Second, the investor optimally pays a more efficient manager more cash for his better skill and incentivizes a less efficient manager with the performance of his trading in the stock. The optimal compensation contains cash and an active benchmark portfolio. Both the cash and the stock holding in the benchmark portfolio are higher for a more efficient manager. A less efficient manager outperforms his *lower* benchmark and obtains a higher incentive pay in bull markets. Third, the time pattern of stock investment can exhibit non-monotone behavior. An efficient manager's investment in the stock decays through time whereas an inefficient manager's trading decreases initially and then increases near the investment horizon. In addition, we make two methodological contributions. First, we show that, under certain conditions, there exists an optimal *linear* contract which consists of the terminal wealth of the portfolio minus a *market-based* component. Second, we show how to design a linear contract which induces the manager to truthfully report his type and adopt a trading policy which is optimal for the reported type.

JEL CLASSIFICATION: D86; C61; G23; G34.

Key words: Portfolio delegation; Adverse selection; Hidden skill; Optimal compensation.

*Department of Finance, School of Business and Management, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong, E-mail: nengjiu@ust.hk; and SAIF, Shanghai Jiaotong University, China, email: njju@saif.sjtu.edu.cn.

†Department of Information and Systems Management, School of Business and Management, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong. Phone: (852) 2358-7731. E-mail: imwan@ust.hk.

Delegated Portfolio Management under Adverse Selection in a Continuous-Time Model

ABSTRACT

This paper studies the optimal contracting problem between a representative fund investor and a constant absolute risk-aversion fund manager whose skill is unobservable by the investor. The optimal compensation contract and the manager's optimal fund investment policy are derived in closed form. Several interesting results are obtained. First, the optimal contract involves an incentive fee which is symmetric around a market-based benchmark, complementing existing result on the optimality of symmetric incentive fees when the investor and the manager have symmetric information and justifying the use of symmetric incentive fees even when there is asymmetric information. Second, the investor optimally pays a more efficient manager more cash for his better skill and incentivizes a less efficient manager with the performance of his trading in the stock. The optimal compensation contains cash and an active benchmark portfolio. Both the cash and the stock holding in the benchmark portfolio are higher for a more efficient manager. A less efficient manager outperforms his *lower* benchmark and obtains a higher incentive pay in bull markets. Third, the time pattern of stock investment can exhibit non-monotone behavior. An efficient manager's investment in the stock decays through time whereas an inefficient manager's trading decreases initially and then increases near the investment horizon. In addition, we make two methodological contributions. First, we show that, under certain conditions, there exists an optimal *linear* contract which consists of the terminal wealth of the portfolio minus a *market-based* component. Second, we show how to design a linear contract which induces the manager to truthfully report his type and adopt a trading policy which is optimal for the reported type.

JEL CLASSIFICATION: D86; C61; G23; G34.

Key words: Portfolio delegation; Adverse selection; Hidden skill; Optimal compensation.

1. Introduction

The percentage of household wealth managed by financial institutions has greatly increased in recent years. In 2002, around 96% of New York Stock Exchange trading volumes were non-retail, according to Jones and Lipson (2004). In fund-based asset management, most investment decisions are delegated to fund managers whose skill and trading behavior are imperfectly observed. Kacperczyk, Sialm and Zheng (2008) analyze the impact of unobserved actions on fund performance empirically using a large sample of U.S. equity mutual funds from 1984 to 2003 and find that funds differ substantially with respect to the impact of such actions. Baik, Kang and Kim (2010) investigate the empirical evidence supplying a link between managers' informational advantage and fund performance employing geographic proximity as a measure of managers' hidden information. Given the prominent role fund management plays, it is of paramount importance to study how fund managers are incentivized by their compensations subject to asymmetric information.

We investigate the optimal compensation between a representative risk-neutral fund investor and a constant absolute risk aversion (thereafter CARA) fund manager in a continuous-time agency model. In the model, the manager's skill is only known by the manager himself and hidden to the investor (thus a hidden skill model). The investor offers a manual of contracts for each skill reported by the manager.[1] The investor's optimal contract and the manager's optimal portfolio investment policy are derived in closed form.[2] Using our closed form solutions, we obtain a number of interesting results.

First, the optimal contract involves an incentive fee which is symmetric around a market-based component.[3] We find that this market-based component can be further decomposed into cash, managing cost and a benchmark portfolio. Consequently, the optimal compensation contract consists of cash, managing cost, plus a bonus or a penalty depending upon the *difference* between the value of the managed portfolio and that of the benchmark portfolio. All of these components are contingent upon the skill reported by the manager.

[1] Effectively, the manager chooses which contract to take by his choice of what skill to report. We will require and design the contact that induces the manager to truthfully report his skill.

[2] Edmans and Gabaix (2011) make an important advance in developing tractable optimal contracts. However, their general framework does not apply to a delegated portfolio management problem.

[3] By market-based, we mean that it depends only on the market condition (i.e. the evolution of the stock price), not on the manager's trading policy (i.e. managerial action).

Ou-Yang (2003) has economically justified the use of symmetric incentive fee when the investor and the manager have symmetric information. Our result not only complements his result but extends the existing result when there is asymmetric information, thus providing a stronger economic justification for the requirement of the Amendment to the Investment Advisors Act of 1940 in the US that performance-based incentive fees involving a market benchmark be symmetric around the chosen benchmark.

Second, a more efficient manager is compensated with more cash and a less efficient manager is incentivized with his outperformance over the benchmark portfolio. A more efficient manager trades more in the stock since his trading cost is lower. However, the optimal incentive contract calls for a higher holding of the stock in the benchmark portfolio. It turns out that the *difference* between the manager's trading of the stock and the benchmark portfolio holding is lower (higher) for a more (less) efficient manager. Thus, even though a less efficient manager trades less actively in the stock, the benchmark holding is optimally lowered to the extent that he is incentivized more with his trading in the stock. To sum, the investor pays an efficient manager more cash for better skill and incentivizes a less efficient manager with the performance from his trading in the the stock.

Aggarwal and Jorin (2010) find that managers of emerging hedging funds tend to add value in their early years and fund values decay in the later stages. The finding is explained by a very significant backfill bias from the perspective of implicit incentives. This motivates us to investigate the time pattern of the manager's trading. Note that backfill bias is excluded by the setup of our model. Hence, time-pattern of the manager's trading is purely affected by manager's explicit incentives driven by the compensation contract.

As our third major result, the time pattern of stock investment can exhibit non-monotone behavior. An efficient manager's investment in the stock decays through time whereas an inefficient manager's trading decreases initially and then increases near the investment horizon. There is a natural mechanism (the interest rate effect) that reduces the manager's stock holding in the fund through time.[4] However, a less efficient manager's trading behavior can be different. When the xxx time approaches the investment horizon, the agency conflict for an inefficient manager decreases. The reason is that an inefficient manager has a higher incentive component in

[4]See, e.g., $\pi_F(t,\delta)$ in (8) where $h(t) = e^{r(T-t)}$ and r is the interest rate, T the terminal date and t the current time. When $r > 0$, $\pi_F(t,\delta)$ is decreasing in t.

his compensation and prefers to taking higher risk in the short remaining period for the hope of good stock performance. Hence, near the contract horizon, the decrease of agency cost dominates the interest rate effect and the inefficient manager's investment in stock increases.

We also make two methodological contributions. First, we prove a general and important result that, under the space of hedge-proof contracts, *optimal linear* contracts exist (Proposition 1). Hedge-proof contracts require that the manager do not trade the same stock (the stock he invests in the fund) in his personal portfolio. The reason is that if the manager invests the same stock in his personal portfolio, then his stock investment in the fund will be affected since the risk in compensation contract can be undone (hedged) by trading the same stock in the manager's personal portfolio.[5] Thus, we require that, given the contract, the manager optimally choose not to trade the stock in his personal portfolio. The importance of the proposition is that we only need to determine the *optimal* contract from the space of *linear* contracts.

Proposition 1 only ensures the existence of optimal hedge-proof linear contracts. As our second methodological contribution, we show how to design (determine) such linear contracts (Theorem 3) which induces the manager not to hedge willingly, to report his skill truthfully and trade optimally. Although it is standard in the agency literature to assume that the manager does not trade in his personal portfolio, we *endogenize* this requirement by properly designing the compensation contract so that the manager willingly (i.e. optimally) chooses not to trade in his personal portfolio.

Our model development differs in several aspects from most extant literature. Our model is dynamic and places no restriction on the contract form whereas most models concerning the optimal contract for delegated investment are static and/or assume specific contract forms.[6,7] Dynamic delegated portfolio management also differs from features of standard agency models. For example, portfolio managers influence both the level and volatility of the portfolio returns whereas managers in standard agency models such as Holmstrom and Milgrom

[5] For more discussion on this assumption/requirement, see Cuoco and Kaniel (2011), P. 272.

[6] A partial list includes, among others, Stoughton (1993), Li and Tiwari (2009), He and Xiong (2010), Kyle, Ou-Yang and Wei (2010), Dybvig, Farnswork and Carpenter (2010).

[7] Although we consider only linear contracts, Proposition 1 shows that they are optimal under the space of hedge-proof contracts.

(1987) and Schattler and Sung (1993) affect level only.[8] Furthermore, the compensation cannot be contingent upon the path of the output.[9] Otherwise, quadratic variation reveals the manager's investment decision at all times. Also importantly, a fund manager's compensation can include market benchmark whereas it is lacking in standard agency models.

The rest of the paper is organized as follows. The next section specifies the setup and assumptions of the hidden skill model. Section 3 presents the contracting problems and their closed form solutions. Section 4 examines the major properties of the solutions. Section 5 extends the models in Section 3 with a risk-neutral investor to a CARA investor. Section 6 concludes. The appendices contain the technical derivations and proofs.

2. The Model Setup

A principal hires an agent to manage his wealth. Similar to Ou-Yang (2003), the principal is the representative investor of a mutual or pension fund and the agent is the fund manager who represents the fund company.[10] Our primary focus is on optimal contracting under portfolio delegation with adverse selection. For analytical tractability the principal-agent model is cast in continuous-time over the time horizon $[0,T]$.

2.1. Investment Opportunity Set

Assumption 1 *There are two assets available for the agent to trade: one riskless bond with a constant return r and a non-dividend paying risky stock $S(t)$ whose dynamics follow a geometric Brownian motion,*[11]

$$dS(t) = S(t)\left(\mu dt + \sigma dZ(t)\right), \tag{1}$$

where μ and σ are constants and $Z(t)$ is a Wiener process on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

[8] More recent work in which managerial action affects only level includes, among others, Sannikov (2008), He (2009, 2011), Cvitanić, Wan and Zhang (2009), Ju and Wan (2011).

[9] See Sung (2005) for an example where the contract depends on the whole sample path.

[10] We will use principal and investor, agent and manager interchangeably.

[11] To simplify notation, we make the innocuous assumption of only one risky asset. Multiple assets can be handled straightforwardly.

Let $\mathcal{F}_t$ denote the filtration generated by $Z(t)$. Equivalently, $\mathcal{F}_t$ denotes the filtration generated by $S(t)$. Conditional expectation under $\mathbb{P}$ is denoted as $\mathbb{E}[\cdot|\mathcal{F}_t]$. To save notation, unconditional expectation $\mathbb{E}[\cdot|\mathcal{F}_0]$ will be simply denoted as $\mathbb{E}[\cdot]$.

Let $\pi(t)$ denote the portfolio policy (dollar investment) in the risky stock at time t in the fund by the manager. The fund wealth process $W^{\pi}(t)$ follows

$$dW^{\pi}(t) = (rW^{\pi}(t) + \eta\pi(t))\,dt + \sigma\pi(t)dZ(t), \qquad W^{\pi}(0) = W_0, \tag{2}$$

where $\eta = \mu - r$ and W_0 is the initial investment (capital) in the fund. In the following we often need to use $W^{\pi}(T)$. To this end, define

$$h(t) = e^{r(T-t)}. \tag{3}$$

It is easy to check that

$$d\left(h(t)W^{\pi}(t)\right) = h(t)\eta\pi(t)dt + h(t)\sigma\pi(t)dZ(t). \tag{4}$$

Therefore,

$$W^{\pi}(T) = e^{rT}W_0 + \int_0^T h(t)\eta\pi(t)dt + \int_0^T h(t)\sigma\pi(t)dZ(t). \tag{5}$$

Besides studying the optimal portfolio delegation under adverse selection, as benchmarks, we will also discuss the first-best and second-best optimal contracting and portfolio policies and the benchmark portfolio under adverse selection. For easy reference of different notations, we list them in the following table.

Portfolio	Investment policy	Wealth at T	Initial value
Generic	π	$W^{\pi}(T)$	W_0
First-best fund	π_F	$X^{\pi_F}(T)$	W_0
Second-best fund	π_S	$X^{\pi_S}(T)$	W_0
Fund under adverse selection	π_D	$X^{\pi_D}(T)$	W_0
Benchmark under adverse selection	ν_D	$X^{\nu_D}(T)$	W_0

2.2. Economic Agents

Assumption 2 *The principal is risk-neutral and the agent exhibits CARA with coefficient $R_A > 0$. There is no intermediate consumption. The agent's monetary cost rate associated with trading in the stock is modeled as a standard quadratic function,*

$$g(\delta, \pi(t)) = \frac{\delta}{2}\pi^2(t), \tag{6}$$

where δ is the manager's skill (efficiency) or type.[12]

We consider a CARA principal in Section 5 and show that our major results with a risk-neutral principal still hold. Therefore, we will focus our discussions with a risk-neutral principal.

Assumption 3 *The investor has imperfect knowledge of the manager's skill or ability δ.*[13] *However, the manager knows his own skill.*

The investor's prior is that δ is a random variable independent of all other random variables in this paper and is distributed on a compact interval $\Theta = [\underline{\delta}, \bar{\delta}]$ with $\underline{\delta} > 0$. Let $f(\delta)$ and $F(\delta)$ denote the probability density function and the distribution function, respectively.

2.3. The Contract Space

Assumption 4 *The investor can observe only fund terminal value $W(T)$ and the path of $S(t)$. At time 0, the investor offers a menu of contracts (one for each reported δ) which are contingent only on the investor's*

[12] This cost function is adopted from Ou-Yang(2003). It is implicitly assumed that the more the manager invests in the risky stock, the greater the effort he must expend in acquiring information about it and monitoring its price movement. Managing costs are likely to change and time varying (Fama and French (2010)).

[13] A long-standing puzzle is the underperformance of actively managed funds. This casts doubts about the existence of smart fund managers with good ability. Elton, Gruber, Das and Hlavka (1993) find that actively managed funds, on average, provide lower net returns than passively managed indexes. Grinblatt and Titman (1989,1993) and Wermers (2000) try to explain this underperformance puzzle by pointing out that actively managed funds may outperform their benchmarks before expenses are deducted. Baks, Metrick and Wachter (2001), Kosowski, Timmermann, Wermers and White (2006) and Avramov and Wermers (2006) uncover the existence of outperforming funds. Barras, Scaillet and Wermers (2010) show that the proportion of skilled fund managers has diminished rapidly over the past 20 years and argue that the long-standing puzzle of actively managed mutual fund underperformance is due to the long-term survival of a minority of truly underperforming funds and hence there do exist managers with more information or better ability.

observables and the skil reported by the manager. We require that the contract induce the manager to truthfully report his skill in equilibrium and optimally not trade the same stock in his personal portfolio (i.e. the manager does not hedge risk in his compensation by trading the stock in his personal portfolio). We call such a contract a hedge-proof contract and denote it by $C(T,\delta,W(T),S(.))$.

2.3.1. An Irrelevance Result for the Contract Space (Form)

In general, $C(T,\delta,W(T),S(.))$ is nonlinear in all entries. In the following proposition, we show that among all hedge-proof contracts, there exists a linear contract which can enforce the agent to report truthfully his skill and choose the same fund trading policy and not trade in his personal portfolio. Due to this "Irrelevance Result," without loss of generality, we restrict our contract space to linear contracts, as long as we require that the contracts be hedge-proof (i.e., the agent optimally chooses not to trade the stock in his personal portfolio).

Proposition 1 ***(Irrelevance Result)*** *Assume that* $C(T,\delta,W(T),S(.))$ *can enforce the agent to report his skill* δ *truthfully, choose* $\pi(t,\delta)$ *optimally and not to hedge privately. Then, there exists a market-based term* $B(T,\delta,S(.))$ *such that the new contract* $C^a(T,\delta,W(T),S(.)) = W(T) - B(T,\delta,S(.))$ *can also enforce the agent to truthfully report* δ*, optimally choose* $\pi(t,\delta)$ *and not hedge privately. Moreover, at the equilibrium,*

$$C^a\left(T,\delta,W^{\pi(.,\delta)}(T),S(.)\right) = C\left(T,\delta,W^{\pi(.,\delta)}(T),S(.)\right). \tag{7}$$

That is, $C^a(T,\delta,W(T),S(.))$ *is equivalent to* $C(T,\delta,W(T),S(.))$

Proof: See Appendix A.

This proposition holds even if the agent has more general preferences and cost functions. The irrelevance result implies that, instead of finding an optimal contract in a very complex contract space, we can consider only the compensation that takes the form $W(T) - B(T,\delta,S(.))$. Stoughton (1993) and Amati and Pfleiderer (1997) show that if the manager can costlessly invest in the capital market, then the market benchmark provides no incentive. This is referred to as the "irrelevance result." In this paper, the investment is costly and we have our version of "irrelevance result" with the assumption that the manager can hedge but the contract induces him not to. The market-based benchmark $B(T,\delta,S(.))$ is the unique instrument providing the incentive.

3. Contracting Problems

Before we present our main problem which we call it the third-best problem, as benchmarks and for comparisons, we introduce the first-best and second-best compensation and portfolio management problems. These problems are defined in the following table.[14]

Type of problem	Who makes investment policy?	What investor knows of manager type?
First-best	Investor dictates $\pi_F(t,\delta)$	Investor knows manager type δ
Second-best	Investor dictates $\pi_S(t,\delta)$	Investor does not know δ, but knows its density $f(\delta)$
Third-best	Manager chooses $\pi_D(t,\delta)$	Investor does not know δ, but knows its density $f(\delta)$

3.1. The First-Best Problem

The principal knows the manager's skill δ and observes, i.e. can dictate the manager's trading policy $\pi(t,\delta)$ in the fund. The principal determines the first-best *fee* $P(T,\delta,S(.))$ and portfolio policy $\pi(t,\delta)$ to maximize

$$\mathbb{E}\left[W^{\pi(.\delta)}(T) - P(T,\delta,S(.))\right],$$

subject to the manager's individual reservation constraint (IR)

$$(IR) \qquad \mathbb{E}\left[-\exp\left\{-R_A\left[P(T,\delta,S(.)) - \int_0^T \frac{\delta}{2}\pi^2(s,\delta)\,ds\right]\right\}\right] \geq -\exp\left\{-R_A\overline{U}\right\}.$$

[14] δ in the second-best and third-best investment policies, $\pi_S(t,\delta)$ and $\pi_D(t,\delta)$, is the manager's *report* of his type.

We will show that the first-best *fee* and optimal portfolio policy can be achieved through a *contract* without dictation by the investor.[15] Therefore, in this setting (no uncertainty on manager's skill δ), the first-best (i.e. with dictation) and second-best (i.e. without dictation) solutions coincide. For this reason, we call both solutions with no uncertainty on manager's skill the first-best solution and refer the second-best to the setting where the investor does not know the manager's skill but still dictates the trading policy.

We summarize the results in the following theorem.

Theorem 1 *The first-best portfolio investment and the first-best fee are given by*

$$\pi_F(t,\delta) = h(t)\delta^{-1}\eta, \qquad P_F(T,\delta,S(.)) = \overline{U} + \int_0^T \frac{\delta}{2}\pi_F^2(t,\delta)dt. \tag{8}$$

Furthermore, the first-best solution (portfolio policy and fee) can be implemented by the following contract,[16]

$$C_F\left(T,\delta,W\left(T\right),S\left(.\right)\right) = \underbrace{W(T) - X^{\pi_F(.,\delta)}\left(T\right)}_{Excess\ return} + \underbrace{\overline{U}}_{Cash} + \underbrace{\int_0^T \frac{\delta}{2}\pi_F^2\left(t,\delta\right)dt}_{Payment\ of\ Cost}. \tag{9}$$

That is, given contract $C_F\left(T,\delta,W\left(T\right),S\left(.\right)\right)$, the manager's optimal portfolio investment is $\pi_F\left(.\delta\right)$. Moreover, at the equilibrium, i.e. with $\pi_F\left(.\delta\right)$, $C_F\left(T,\delta,W^{\pi_F(.,\delta)}\left(T\right),S\left(.\right)\right) = P_F(T,\delta,S(.))$.

Proof: See Appendix C.[17]

Remark 1 *$\pi_F(t,\delta)$ is known to the investor and $X^{\pi_F(.,\delta)}\left(T\right)$ is the benchmark portfolio wealth at T from policy $\pi_F(t,\delta)$. $W(T)$ is the fund's actual wealth resulting from the manager's (unobserved) policy $\pi\left(t,\delta\right)$. The difference, $W(T) - X^{\pi_F(.,\delta)}\left(T\right)$, measures the manager's performance against the benchmark. At equilibrium, the manager optimally adopts $\pi_F\left(t,\delta\right)$ and $W(T) - X^{\pi_F(.,\delta)}\left(T\right)$ becomes zero.*

[15] We distinguish compensation *fee* and compensation *contract*. The value of *fee* is exogenous, i.e. it does not depend on the agent's action (i.e. investment policy) but the value of contract does.

[16] Technically, we should have used $C_F^a(.)$ to denote the contract (see Proposition 1). To simplify notation, we drop the superscript.

[17] This is the problem solved in Ou-Yang (2003). To provide the notations and develop the solution techniques for our second-best and third-best problems, a complete but different and more concise proof of this first-best problem is still provided.

We note that the optimal first-best *fee* is a constant. This is due to the risk-neutrality of the investor (this comment also applies to the optimal second-best fee which we consider next). However, the optimal *contract* (9) is certainly not a constant. Only in equilibrium is it. Moreover, when we extend to a CARA investor in Section 5, neither the optimal fee nor the optimal contract is a constant, even in equilibrium.

3.2. The Second-Best Problem

In the second-best setting, the principal does not know the manager's skill, but still observes, i.e. dictates the manager's trading, based on manager's *report* of his skill. Effectively, the investor offers a *manual* of fee $P_S(T,\delta,S(.))$ and portfolio trading $\pi_S(t,\delta)$ where δ is the manager's report of his skill. The manager chooses what to receive, $P_S(T,\delta,S(.))$, and what to trade, $\pi_S(t,\delta)$, (both are dictated by the investor), by his choice of what to report, δ. The manager can misreport.

Definition 1 *Truth telling (TT) constraint: The (TT) constraint is satisfied if reporting his skill truthfully is optimal for the manager.*

We design the manual $(\pi_S(t,\delta),P_S(T,\delta,S(.)))$ subject to the (TT) constraint. To ensure the optimality (i.e. the second-order condition) of the resulting solution, we need the following *MHRP* constraint.

Definition 2 *The distribution of* δ *satisfies the "monotone hazard rate property" (thereafter MHRP) if* $\frac{F(\delta)}{f(\delta)}$ *is increasing in* δ*, i.e. if* $\frac{d}{d\delta}\left(\log\left[\frac{F(\delta)}{f(\delta)}\right]\right)\geq 0$.

We are now ready to state the second-best problem. The principal determines the optimal fee $P(T,\delta,S(.))$ and optimal portfolio policy $\pi(t,\delta)$ to maximize

$$\int_{\underline{\delta}}^{\bar{\delta}}\mathbb{E}\left[W^{\pi(.,\delta)}(T)-P(T,\delta,S(.))\right]f(\delta)d\delta, \qquad (10)$$

subject to the manager's individual reservation constraint (IR) and the (TT) constraint,

$$(TT) \qquad \delta \in \arg\max_{\hat{\delta}\in\Theta} \mathbb{E}\left[-\exp\left\{-R_A\left[P(T,\hat{\delta},S(.)) - \int_0^T \frac{\delta}{2}\pi^2\left(s,\hat{\delta}\right)ds\right]\right\}\right]. \tag{11}$$

Note that in (11), given that the manager reports $\hat{\delta}$, he is paid with $P(T,\hat{\delta},S(.))$. His cost is still computed using his true skill δ with the portfolio policy $\pi\left(s,\hat{\delta}\right)$ which is dictated by the principal based on reported $\hat{\delta}$. (TT) states that given the manual, $(\pi(t,\delta),P(T,\delta,S(.)))$, reporting truthfully is optimal for the manager.

Theorem 2 *Assume that the prior distribution of manager's skill satisfies the MHRP. Then the second-best portfolio investment $\pi_S(t,\delta)$ and fee $P_S(T,\delta)$, subject to the (TT) constraint, are given by*

$$\pi_S(t,\delta) = h(t)\left[\delta + \frac{F(\delta)}{f(\delta)}\right]^{-1}\eta, \qquad P_S(T,\delta,S.)) = U_S(0,\delta) + \int_0^T \frac{\delta}{2}\pi_S^2(t,\delta)dt, \tag{12}$$

where $U_S(0,\delta) = \overline{U} + \mathcal{IM}_S(\delta)$ and

$$\mathcal{IM}_S(\delta) = \int_{\delta}^{\bar{\delta}}\int_0^T \frac{1}{2}\pi_S^2(t,\tau)dtd\tau = \frac{e^{2rT}-1}{4r}\eta^2\int_{\delta}^{\bar{\delta}}\left[\delta + \frac{F(\delta)}{f(\delta)}\right]^{-2}d\delta \tag{13}$$

measures the information rent paid to the manager with skill δ and becomes zero for the most inefficient manager with $\delta = \bar{\delta}$.

Proof: See Appendix D.

The second-best portfolio policy is not implementable. That is, there does not exist a contract which induces the manager to optimally adopt the second-best portfolio policy $\pi_S(t,\delta)$ without dictation from the investor. The proof is that the optimal portfolio policy $\pi_D(t,\delta)$ in the third-best problem which we consider next differs from $\pi_S(t,\delta)$. This is different from the situation in Theorem 1 when the investor knows the manager's skill. In such a case the optimal policy without dictation is the same as that with dictation.

3.3. The Third-best Problem

In the third-best setting, the principal does not know the manager's skill and does not dictate the trading policy either. The manager can misreport.

Definition 3 *Truthful enforcement (TE): The (TE) constraint is satisfied if truthfully reporting his skill δ AND adopting portfolio policy which is optimal for a manager with skill δ are jointly optimal for the manager.*

The main problem of this paper is to find the compensation contract $C_D(T,\delta,W(T),S(.))$ that maximizes the investor's expected value AND satisfies the (TE) constraint. To ensure the optimality of the solution, we need the following *SMHRP* constraint.

Definition 4 *The distribution of δ satisfies the "sharp monotone hazard rate property" (thereafter SMHRP) if $\frac{F(\delta)}{(\delta+R_A\sigma^2)^2 f(\delta)}$ is increasing in δ, i.e. if $\frac{d}{d\delta}\left(\log\left[\frac{F(\delta)}{f(\delta)}\right]\right) \geq \frac{2}{\delta+R_A\sigma^2}$.*

Remark 2 *The classical assumption for the distribution of skills in the study of adverse selection is MHRP. SMHRP is obviously stronger than MHRP. For example, a uniform distribution satisfies MHRP because*

$$\frac{d}{d\delta}\left[\log\left(\frac{F(\delta)}{f(\delta)}\right)\right] = \frac{1}{\delta-\underline{\delta}} \geq 0, \qquad \forall\delta \in [\underline{\delta},\bar{\delta}], \tag{14}$$

but SMHRP requires $\bar{\delta}-2\underline{\delta} \leq R_A\sigma^2$.

We now state the third-best problem. The principal chooses contract $C_D(T,\delta,W(T),S(.))$ to maximize

$$\int_{\underline{\delta}}^{\bar{\delta}} \mathbb{E}\left[W(T)-C_D(T,\delta,W(T),S(.))\right] f(\delta)\, d\delta, \tag{15}$$

subject to the individual reservation (IR) and the (TE) constraint,

$$(TE) \qquad (\delta,\pi(t,\delta)) \in \arg\max_{\hat{\delta},\hat{\pi}(.)} \mathbb{E}\left[-\exp\left\{-R_A\left[C_D(T,\hat{\delta},W^{\hat{\pi}}(T),S(.)) - \int_0^T \frac{\delta}{2}\hat{\pi}^2(s)\, ds\right]\right\}\right].$$

(TE) states that, given $C_D(T,\hat{\delta},W^{\hat{\pi}}(T),S(.))$, it is optimal for the manager to report δ truthfully and adopt $\pi(t,\delta)$ which is the optimal policy for type δ.

Theorem 3 ***(Optimal Compensation with Unobserved Skill)*** *Assume that the agent's skill δ is hidden, and the manager's trading cannot be dictated. Furthermore, we assume that the prior distribution of the skill satisfies the SMHRP. Then, subject to the (TE) constraint, the manager's optimal portfolio investment is*

$$\pi_D(t,\delta) = h(t)\left[\delta + \left(\frac{R_A\sigma^2 h^2(t)}{\delta + R_A\sigma^2 h^2(t)}\right)\frac{F(\delta)}{f(\delta)}\right]^{-1}\eta, \tag{16}$$

the optimal compensation contract is

$$C_D(T,\delta,W(T),S(.)) = \underbrace{W^{\pi(.)}(T) - X^{v_D(.,\delta)}(T)}_{Excess\ return} + \underbrace{U_D(0,\delta) - \mathcal{UM}_D(\delta)}_{Cash} + \underbrace{\int_0^T \frac{\delta}{2}\pi_D^2(t,\delta)\,dt}_{Payment\ of\ cost}, \tag{17}$$

where $X^{v_D(.,\delta)}(t)$ is the benchmark portfolio with investment policy

$$v_D(t,\delta) = \pi_D(t,\delta) + \frac{\delta}{R_A h^2(t)\sigma^2}\left[\pi_D(t,\delta) - \pi_F(t,\delta)\right], \tag{18}$$

$U_D(0,\delta) = \overline{U} + \mathcal{IM}_D(\delta)$ where

$$\mathcal{IM}_D(\delta) = \int_\delta^{\bar{\delta}}\int_0^T \frac{1}{2}\pi_D^2(t,\tau)dtd\tau \tag{19}$$

measures the information rent extracted by the manager with skill δ and is zero for the most inefficient manager with $\delta = \bar{\delta}$, and $\mathcal{UM}_D(\delta)$ is the under-performance measure defined by

$$\mathcal{UM}_D(\delta) = \int_0^T\left[\frac{\delta\eta}{R_A\sigma^2 h(t)}\left(\pi_F(t,\delta) - \pi_D(t,\delta)\right) - \frac{\delta^2}{2R_A\sigma^2 h^2(t)}\left(\pi_F(t,\delta) - \pi_D(t,\delta)\right)^2\right]dt \tag{20}$$

which is positive for $\delta < \bar{\delta}$ and is zero (no under-performance) for the most efficient manager with $\delta = \underline{\delta}$.

Proof: See Appendix E.

Our model represents an extension to Ou-Yang(2003) where the investor and the manager have complete and symmetric information. It is easy to see that the solution in Theorem 3 collapses to the first-best one in Theorem 1 when the distribution of δ becomes a singleton, i.e. when $f(\delta)$ becomes infinite at a single point. With symmetric information, both the information rent and the under-performance measure become 0.

4. Discussion of Results

4.1. Form of Optimal Compensation Contract

The optimal compensation contracts in both the first-best and third-best model (recall that there there does not exist an implementable contract in the second-best model) take a symmetric form with an active benchmark portfolio which invests a time-dependent and stochastic number of shares in the risky asset. Whether there exists hidden information or not, contingent on the manager's skill (no hidden information) or reported skill (with hidden information), the investor should pay the manager a fixed fee (cash), the cost of managing the fund, plus a *bonus* or a *penalty* depending upon the excess return (value difference) between the managed fund and a benchmark portfolio. Therefore, the form of our contracts provides a theoretical support for the existing regulation which requires that the incentive fee paid to a fund manager be of the symmetric form.

Furthermore, although it is known that the optimal contract is linear (Theorem 1 and Ou-Yang (2003)) when there is no hidden information, our result of optimal linear contract under hidden information provides a stronger foundation for the use of linear contracts.

4.2. Adverse Selection and Incentive Provision

In this section, we investigate the properties of portfolio policies and how incentive is provided by the benchmark. With simple algebra, we have the following three corollaries.

Corollary 1 *$\pi_F(t,\delta)$ is obviously decreasing in δ. Under the assumption of SMHRP, $\pi_D(t,\delta)$ and $v_D(t,\delta)$ are also respectively decreasing in δ.*

Corollary 2 *Under the assumption of SMHRP, $v_D(t,\delta) \le \pi_S(t,\delta) \le \pi_D(t,\delta) \le \pi_F(t,\delta)$. Furthermore, both $\pi_D(t,\delta) - v_D(t,\delta)$ and $\pi_F(t,\delta) - \pi_D(t,\delta)$ are increasing in δ.*

Corollary 3 *Under the assumption of SMHRP, the under-performance measure is zero when $\delta = \underline{\delta}$ and increasing in δ (thus positive) when $\delta > \underline{\delta}$.*

In Figure 1, the interest rate is zero and the stock holding is constant over time (no interest rate effect because $h(t) = 1$) and a function of type δ only. Figure 1 clearly demonstrates the results in Corollaries 1, 2 and 3. More efficient managers (with smaller δ) have higher information rents and lower under-performance measures with respect to the first-best trading policies. Hence, more efficient managers have higher cash compensations.[18]

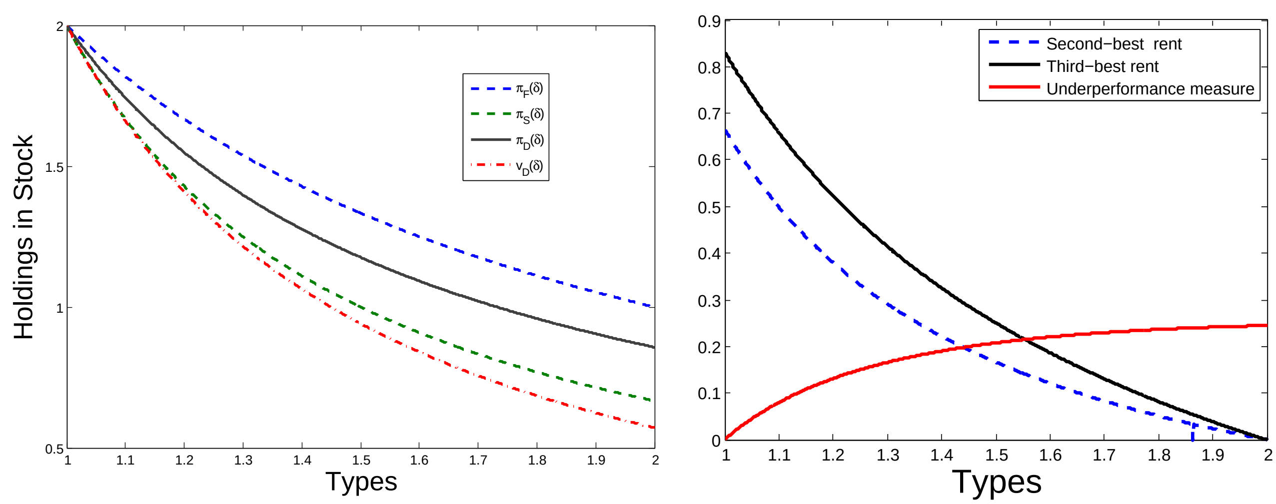

Figure 1
Stock holdings and cash compensation. $T = 1$**,** $r = 0, \mu = 2$**,** $f(\delta) = 1$ **on** $[1,2]$**,** $R_A = \sigma = 1$**,** $\bar{U} = 0$**.**

From the point view of optimal contracting, a benchmark with high holding in stock cannot incentivize an inefficient manager to trade actively in the stock since it is more costly for him to trade. Corollary 1 indicates

[18]Recall from (17) that the cash component is $\overline{U}$ + informational rent - under performance.

that stock trading and benchmark holding are both higher (lower) for a more (less) efficient manager. The investor optimally pays more cash for the good skill of an efficient manager whereas lowers the benchmark holding in stock to incentivize a less efficient manager. However, Corollary 2 shows that, the difference, $\pi_D(t,\delta)-v_D(t,\delta)$, is in fact larger for a less efficient manager. Therefore, even though a less efficient manager has a lower cash compensation, his incentive pay from excess return by trading on the stock (the first part, excess return, in (17)) is higher than that of a more efficient manager if the stock performs well over the time period $[0,T]$.[19] In short, the investor pays more cash for the better skill of an efficient manager and incentivizes an inefficient manager with the performance of his stock trading.

4.3. Time Pattern of Optimal Portfolio Investment

With simple algebra, we have

$$\frac{\partial \pi_D(t,\delta)}{\partial t} = r\eta h(t)\left[\delta + \left(\frac{R_A\sigma^2 h^2(t)}{\delta + R_A\sigma^2 h^2(t)}\right)\frac{F(\delta)}{f(\delta)}\right]^{-2}\left\{\frac{R_A\sigma^2 h^2(t)(\delta - R_A\sigma^2 h^2(t))}{[\delta + R_A\sigma^2 h^2(t)]^2}\frac{F(\delta)}{f(\delta)} - \delta\right\}, \tag{21}$$

It is clear that if the manager is efficient , $\delta < R_A\sigma^2$, his risk-taking decays in time. Even if the manager is not so efficient that $\delta > R_A\sigma^2$, but there exists $t^* < T$ such that $\delta < R_A\sigma^2 h^2(t^*)$,[20] then $\frac{\partial \pi_D(t,\delta)}{\partial t} < 0$ for $t \leq t^*$. That is, if the horizon T is long enough, there exists a t^* such that all managers' trading in stock market decays in earlier stages and then increases for $t > t^*$. A sufficient condition for the existence of a manager with $\delta < R_A\sigma^2 h^2(t^*)$ is that $\bar{\delta} > R_A\sigma^2$ and $\frac{1}{\left[\bar{\delta}+R_A\sigma^2\right]^2 f(\bar{\delta})} > \frac{\bar{\delta}}{R_A\sigma^2(\bar{\delta}-R_A\sigma^2)}$. We summarize the time pattern in the following corollary.

Corollary 4 *An efficient manager's risk-taking (stock trading) tends to decay over time. An inefficient manager tends to take more risk in the later stages of the contract period.*

Next we present a numerical example in which the trading of an inefficient manager decreases sharply in the early stages and increases in the later stages. Assume that $F(\delta)$ is given in the left panel of Figure 2. The

[19]On the other hand, a less efficient manager incurs a larger reduction in his compensation when the stock return is poor.

[20]Recall that $h(t) = e^{r(T-t)}$ which is decreasing in t but increasing in T

trading of an inefficient (efficient) manager is shown in middle (right) panel of Figure 2.

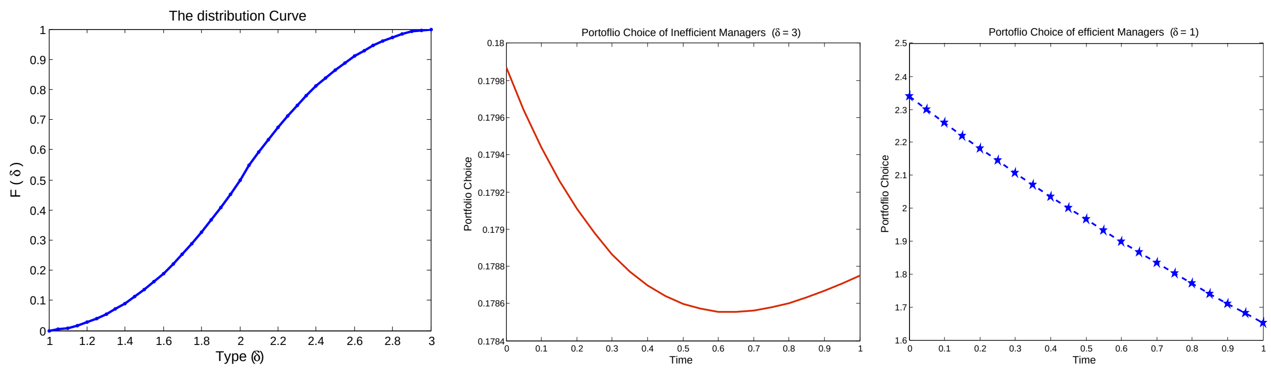

Figure 2
Optimal portfolio investment of the managers. $\bar{\delta} = 3$, $\underline{\delta} = 1$, $R_A = 1$, $\sigma = 1$, $r = 0.3466$, $\mu = 2$.

5. Risk-Averse Investor

In this section, we extend the models with a risk-neutral investor to those with a CARA investor with coefficient R_P. Corresponding theorems 1-3, we have the following theorems 4-6.

Theorem 4 *The first-best portfolio investment and the first-best fee are given by*[21]

$$\pi_F(t,\delta) = \frac{h(t)\eta\delta^{-1}}{1+\frac{R_A R_P}{R_A+R_P}h^2(t)\sigma^2\delta^{-1}}, \tag{22}$$

$$P_F(T,\delta,S(.)) = \overline{U} + \int_0^T \left[\frac{R_A}{2}K_F^2(t,\delta) + \frac{\delta}{2}\pi_F^2(t,\delta)\right] dt + \int_0^T K_F(t,\delta)\, dZ(t), \tag{23}$$

where

$$K_F(t,\delta) = \frac{R_P}{R_A+R_P} h(t)\sigma\pi_F(t,\delta). \tag{24}$$

Furthermore, the first-best solution (portfolio investment and fee) can be implemented by the following contract,

[21] We use the same notations to avoid introducing new ones, even though they are different when the investor is risk-averse.

$$C_F(T,\delta,W(T),S(.)) = \underbrace{W(T) - X^{\widetilde{\pi}_F(.,\delta)}(T)}_{Excess\ return} + \underbrace{\overline{U} + \int_0^T \left[\frac{R_A}{2}K_F^2(t,\delta) - \frac{\eta}{\sigma}K_F(t,\delta)\right]dt}_{Cash} + \underbrace{\int_0^T \frac{\delta}{2}\pi_F^2(t,\delta)\,dt}_{Payment\ of\ Cost}, \tag{25}$$

where $\widetilde{\pi}_F(t,\delta) = \pi_F(t,\delta)R_A/(R_A+R_P)$ *and*

$$X^{\widetilde{\pi}_F}(T) = e^{rT}W_0 + \int_0^T h(t)\eta\widetilde{\pi}_F(t,\delta)dt + \int_0^T h(t)\eta\widetilde{\pi}_F(t,\delta)dZ(t). \tag{26}$$

At the equilibrium, the manager's optimal portfolio investment is $\pi_F(t,\delta)$ *and*

$$C_F\left(T,\delta,W^{\pi_F(.,\delta)}(T),S(.)\right) = P_F(T,\delta,S(.)). \tag{27}$$

Proof: See Appendix F.

Note that when $R_P \neq 0$, the manager's optimal trading $\pi_F(t,\delta)$ is different from the benchmark policy $\widetilde{\pi}_F(t,\delta)$. Moreover, the optimal *fee* is no longer a constant.

Theorem 5 *Assume that the prior distribution of manager's skill satisfies the MHRP. Then, the second-best portfolio investment* $\pi_S(t,\delta)$ *and fee* $P_S(T,\delta)$*, subject to the (TT) constraint, are given by*

$$\pi_S(t,\delta) = h(t)\left[\delta + \frac{R_A R_P}{R_A+R_P}h^2(t)\sigma^2 + \frac{\xi(\delta)}{\xi'(\delta)}\right]^{-1}\eta, \tag{28}$$

$$P_S(T,\delta,S.)) = U_S(0,\delta) + \int_0^T \frac{\delta}{2}\pi_S^2(t,\delta)dt + \int_0^T \frac{R_A}{2}K_S^2(t,\delta)dt + \int_0^T K_S(t,\delta)dZ(t), \tag{29}$$

where

$$U_S(0,\delta) = \overline{U} + \int_\delta^{\bar{\delta}}\int_0^T \frac{1}{2}\pi_S^2(t,\tau)dtd\tau, \qquad K_S(t,\delta) = \frac{R_P}{R_A+R_P}h(t)\sigma\pi_S(t,\delta), \tag{30}$$

and $\xi(\delta)$ *is determined by*

$$U_S(0,\delta) = \frac{1}{R_P}\ln\frac{\xi'(\delta)}{R_P f(\delta)} + \int_0^T H_S(t,\delta,\pi(t,\delta),K(t,\delta))\,dt, \qquad \xi(\underline{\delta}) = 0, \tag{31}$$

where

$$H_S(t,\delta,\pi(t,\delta),K(t,\delta)) = h(t)\,\eta\pi(t,\delta) - \frac{R_A}{2}K^2(t,\delta) - \frac{\delta}{2}\pi^2(t,\delta) - \frac{R_P}{2}(h(t)\,\sigma\pi(t,\delta) - K(t,\delta))^2. \tag{32}$$

Proof: See Appendix G.

Theorem 6 *Assume that the prior distribution of manager's skill satisfies the SMHRP. Then, subject to the (TE) constraint, the third-best portfolio investment $\pi_D(t,\delta)$ is given by*

$$\pi_D(t,\delta) = h(t)\left[\delta + \frac{R_A R_P}{R_A + R_P}h^2(t)\sigma^2 + \left(\frac{R_A\sigma^2 h^2(t)}{\delta + R_A\sigma^2 h^2(t)}\right)\frac{\xi(\delta)}{\xi'(\delta)}\right]^{-1}\eta, \tag{33}$$

the optimal compensation contract that truthfully enforces $\pi_D(t,\delta)$ is given by

$$C_D(T,\delta,W(T),S(.)) = \underbrace{W^{\pi(.)}(T) - X^{v_D(.,\delta)}(T)}_{Excess\ return} + \underbrace{U_D(0,\delta) - \mathcal{UM}_D(\delta)}_{Cash} + \underbrace{\int_0^T \frac{\delta}{2}\pi_D^2(t,\delta)\,dt}_{Payment\ of\ cost}, \tag{34}$$

where

$$U_D(0,\delta) = \overline{U} + \int_\delta^{\bar{\delta}}\int_0^T \frac{1}{2}\pi_D^2(t,\tau)\,dt\,d\tau, \tag{35}$$

$X^{v_D(.,\delta)}(t)$ is the benchmark portfolio with portfolio policy

$$v_D(t,\delta) = \pi_D(t,\delta) + \frac{\delta}{R_A h^2(t)\sigma^2}\left[\pi_D(t,\delta) - h(t)\eta\delta^{-1}\right], \tag{36}$$

$\mathcal{UM}_D(\delta)$ is the under-performance measure defined by

$$UM_D(\delta) = \int_0^T \left[\frac{\delta\eta}{h(t)\,\sigma^2 R_A}\left(h(t)\eta\delta^{-1} - \pi_D(t,\delta)\right) - \frac{\delta^2}{2h^2(t)\,\sigma^2 R_A}\left(h(t)\eta\delta^{-1} - \pi_D(t,\delta)\right)^2\right]dt, \tag{37}$$

and $\xi(\delta)$ is determined by

$$U_D(0,\delta) = \frac{1}{R_P}\ln\frac{\xi'(\delta)}{R_P f(\delta)} + \int_0^T H_D(t,\delta,\pi(t,\delta),K(t,\delta))\,dt, \qquad \xi(\underline{\delta}) = 0, \tag{38}$$

where

$$H_D(t,\delta,\pi(t,\delta),K(t,\delta)) = h(t)\eta\pi(t,\delta) - \frac{R_A}{2}(h(t)\sigma\pi(t,\delta) - K(t,\delta))^2 - \frac{\delta}{2}\pi^2(t,\delta) - \frac{R_P}{2}K^2(t,\delta)). \tag{39}$$

Proof: See Appendix H.

Remark 3 *The solutions (optimal portfolio policies and compensation contracts) are essentially in closed form and similar to those when the investor is risk-neutral, even though the Lagrange multiplier $\xi(\delta)$ is not in closed form when the manager's skill is unobservable by the investor.*

Remark 4 *By (36), we can conclude that, the manager trades more actively if the benchmark holding in stock is higher. Furthermore, the benchmark holding in stock is lower than the manager's optimal stock holding because $\pi_D(t,\delta) \le h(t)\eta\delta^{-1}$. That is, $\pi_D(t,\delta) - v_D(t,\delta)$ is nonnegative for any $\delta \in [\underline{\delta},\bar{\delta}]$. It can be easily checked that both $\pi_D(t,\delta) - v_D(t,\delta)$ and the under-performance measure $UM_D(\delta)$ are increasing in δ. These are consistent with those when the investor is risk neutral.*

Remark 5 *The results that a more efficient manager is compensated more by cash for his better skill and a less efficient manager is incentivized by his trading performance over a benchmark portfolio obtained when the investor is risk-neutral still hold if the investor is risk averse.*

6. Conclusion

This article studies optimal delegated portfolio management under adverse selection in a continuous-time agency model. The skill of the manager is known only to the manager himself and unobservable to the fund investor. The optimal compensation and fund investment policy are derived in closed form. It is found that the optimal compensation involves an incentive fee which is symmetric around a market-based benchmark, thus justifying the use of symmetric incentive fees even when there is asymmetric information. It is also found that the investor optimally pays a more efficient manager more cash for his better skill and incentivizes a less efficient manager with the performance of his trading in the stock. A manager's trading pattern can differ depending on his skill. An efficient manager's investment in the stock decays through time whereas an inefficient manager's trading decreases initially and then increases near the investment horizon.

We have introduced the concept of hedge-proof contracts. Hedge-proof requires the contract to induce the manager willingly not to trade the stock in his personal portfolio, truthfully to report his skill and optimally to invest the stock in the fund. We have shown the existence of equivalent optimal linear contracts under the space of hedge-proof contracts and how to deign (determine) optimal hedge-proof linear contracts.

In this paper, all managers are price-takers. The capital market equilibrium implications of delegated portfolio management is not considered.[22] Wealth effect can be potentially important in the study of portfolio delegation but it is lacking in our use of CARA utility. We have assumed a finite horizon and ignored the manager's career concern. We leave all these for future research.

[22]See Cuoco and Kaniel (2011) for a recent contribution in this area.

Appendices: Technical Proofs and Derivations.

A. Proof of Proposition 1

We prove this proposition with a general cost function $g(\delta, \pi(t))$ and a general utility of the manager $U(.)$. Suppose that $C(T,\delta,W(T),S(.))$ can *truthfully* (i.e. reporting true skill δ) enforce the manager to *optimally* choose portfolio policy $\pi(t,\delta)$ and not to hedge *privately*. Define

$$B(T,\delta,S(.)) = W^{\pi(.,\delta)}(T) - C\left(T,\delta,W^{\pi(.,\delta)}(T),S(.)\right),$$

where $W^{\pi(.,\delta)}(T)$ is the terminal fund value resulting from a *benchmark* investment policy $\pi(t,\delta)$. Given that $\pi(t,\delta)$ is the benchmark policy, $B(T,\delta,S(.))$ is not controllable by the manager and depends only on the path of stock price (i.e. market conditions). The new contract is defined as

$$C^a(T,\delta,W(T),S(.)) = W(T) - B(T,\delta,S(.)),$$

where $W(T)$ is the terminal wealth of fund and controlled by the manager's investment policy. We show that $C^a(T,\delta,W(T),S(.))$ can also *truthfully* enforce the manager to *optimally* choose portfolio investment $\pi(t,\delta)$ and not to hedge *privately*.

We prove the proposition by contrapositive. Suppose that, given the new contract $C^a(T,\delta,W(T),S(.))$, the optimal choice for the manager with true skill δ is to report his skill as $\hat{\delta}$, choose fund portfolio policy $\hat{\pi}(t,\delta,\hat{\delta})$ and personal (i.e. hedging) portfolio policy $\hat{\gamma}(t,\delta,\hat{\delta})$. The personal portfolio consists of two *possible* parts: the hedging part which invests in the underlying risky stock and the rest of the portfolio which does not invest in the risky stock. Without loss of generality, assume that the hedging part has an initial zero cost.[23] Let $X^{\hat{\gamma}(.\delta,\hat{\delta})}(T)$ denote the value at time T of the hedging part and $\hat{Y}(T)$ the value of the rest of the personal portfolio, and $Y(T)$

[23]That is, investment in the stock is financed by borrowing the riskless bond.

the value of the personal portfolio when there is no hedging (i.e. no investment in the risky stock).[24] Then, we have

$$\mathbb{E}\left[U_a\left(C^a\left(T,\hat{\delta},W^{\hat{\pi}(.,\delta,\hat{\delta})}(T),S(.)\right)+X^{\hat{\gamma}(.\delta,\hat{\delta})}(T)+\hat{Y}(T)-\int_0^T g\left(\delta,\hat{\pi}(s,\delta,\hat{\delta})+\hat{\gamma}(s,\delta,\hat{\delta})\right)ds\right)\right]$$
$$>\mathbb{E}\left[U_a\left(C^a\left(T,\delta,W^{\pi(.,\delta)}(T)\right)+Y(T)-\int_0^T g\left(\delta,\pi(s,\delta)\right)ds\right)\right], \tag{A.1}$$

where $W^{\hat{\pi}(.,\delta,\hat{\delta})}(T)$ and $X^{\hat{\gamma}(.\delta,\hat{\delta})}(T)$ are the portfolio values resulting portfolio policies $\hat{\pi}(t,\delta,\hat{\delta})$ and $\hat{\gamma}(t,\delta,\hat{\delta})$, respectively. Initially, $W^{\hat{\pi}(.,\delta,\hat{\delta})}(0)=W_0$, $X^{\hat{\gamma}(.,\delta,\hat{\delta})}(0)=0$.

Note that

$$C^a\left(T,\hat{\delta},W^{\hat{\pi}(.,\delta,\hat{\delta})}(T),S(.)\right)=W^{\hat{\pi}(.,\delta,\hat{\delta})}(T)-B(T,\hat{\delta},S(.))=$$
$$W^{\hat{\pi}(.,\delta,\hat{\delta})}(T)-W^{\pi\left(.,\hat{\delta}\right)}(T)+C\left(T,\hat{\delta},W^{\pi\left(.,\hat{\delta}\right)}(T),S(.)\right), \tag{A.2}$$
$$C^a(T,\delta,W^{\pi(.,\delta)}(T),S(.))=W^{\pi(.,\delta)}(T)-B(T,\delta,S(.))=C\left(T,\delta,W^{\pi(.,\delta)}(T),S(.)\right). \tag{A.3}$$

Then (A.1) can be written as

$$\mathbb{E}\left[U_a\left(\begin{array}{c}W^{\hat{\pi}(.,\delta,\hat{\delta})}(T)-W^{\pi\left(.,\hat{\delta}\right)}(T)+C\left(T,\hat{\delta},W^{\pi\left(.,\hat{\delta}\right)}(T),S(.)\right)\\+X^{\hat{\gamma}(.\delta,\hat{\delta})}(T)+\hat{Y}(T)-\int_0^T g(\delta,\hat{\pi}(s,\delta,\hat{\delta})+\hat{\gamma}(s,\delta,\hat{\delta}))ds\end{array}\right)\right]$$
$$>\mathbb{E}\left[U_a\left(C\left(T,\delta,W^{\pi(.,\delta)}(T),S(.)\right)+Y(T)-\int_0^T g\left(\delta,\pi(s,\delta)\right)ds\right)\right]. \tag{A.4}$$

Since

$$X^{\hat{\gamma}(.\delta,\hat{\delta})}(T)+W^{\hat{\pi}(.,\delta,\hat{\delta})}(T)-W^{\pi\left(.,\hat{\delta}\right)}(T)=X^{\hat{\gamma}(.\delta,\hat{\delta})+\hat{\pi}(.,\delta,\hat{\delta})-\pi\left(.,\hat{\delta}\right)}(T),$$
$$\hat{\pi}(s,\delta,\hat{\delta})+\hat{\gamma}(s,\delta,\hat{\delta})=\left[\hat{\gamma}(s,\delta,\hat{\delta})+\hat{\pi}(s,\delta,\hat{\delta})-\pi\left(s,\hat{\delta}\right)\right]+\pi\left(s,\hat{\delta}\right),$$

[24] Note that $\hat{Y}(T)$ and $Y(T)$ can differ even if their initial values are the same because the initial value of the hedging part is zero.

where $\pi\left(s,\hat{\delta}\right)$ denotes the optimal trading policy for type $\hat{\delta}$, we have

$$\mathbb{E}\left[U_a\left(\begin{array}{c} X^{\hat{\gamma}(.\delta,\hat{\delta})+\hat{\pi}(.,\delta,\hat{\delta})-\pi\left(.,\hat{\delta}\right)}(T)+\hat{Y}(T)+C\left(T,\hat{\delta},W^{\pi\left(.,\hat{\delta}\right)}(T),S(.)\right) \\ -\int_0^T g\left(\delta,\left[\hat{\gamma}(s,\delta,\hat{\delta})+\hat{\pi}(s,\delta,\hat{\delta})-\pi\left(s,\hat{\delta}\right)\right]+\pi\left(s,\hat{\delta}\right)\right)ds \end{array}\right)\right]$$
$$>\mathbb{E}\left[U_a\left(C\left(T,\delta,W^{\pi(.,\delta)}(T),S(.)\right)+Y(T)-\int_0^T g(\delta,\pi(s,\delta))ds\right)\right]. \tag{A.5}$$

That is, if the manager's true skill is δ, then given the compensation $C(T,\delta,W(T),S(.))$, he is better off if he chooses to report $\hat{\delta}$, trade $\pi\left(t,\hat{\delta}\right)$ in fund and invest $\hat{\gamma}(t,\delta,\hat{\delta})+\hat{\pi}(t,\delta,\hat{\delta})-\pi\left(t,\hat{\delta}\right)$ in personal hedging portfolio and the rest in $\hat{Y}(T)$. This obviously contradicts with the fact that $C\left(T,\delta,W^{\pi(.,\delta)}(T),S(.)\right)$ enforces: (a) *truthfully* reporting δ; (b) *optimally* adopting $\pi(t,\delta)$; and (c) not hedging *privately*. It follows that, if $C(T,\delta,W(T),S(.))$ can enforce (a), (b) and (c), then

$$C^a(T,\delta,W(T),S(.))=W(T)-B(T,\delta,S(.))=W(T)-\left(W^{\pi(.,\delta)}(T)-C\left(T,\delta,W^{\pi(t,\delta)}(T),S(.)\right)\right)$$

can do the same and satisfies $C^a\left(T,\delta,W^{\pi(.,\delta)}(T),S(.)\right)=C\left(T,\delta,W^{\pi(t,\delta)}(T),S(.)\right)$ in equilibrium.

A.1. A linear Contract Is Necessarily Weakly Hedge-Proof

To better understand the proposition, we show that a linear contract, $C^a(T,\delta,W^{\pi(.,\delta)}(T)=W^{\pi(.,\delta)}(T)-B(T,\delta)$, is necessarily weakly hedge-proof. Let $(\gamma(t,\delta),\pi(t,\delta))$ denote the hedging policy in the personal portfolio and investment policy in the managed fund. The manager's utility becomes

$$\mathbb{E}\left[U_a\left(W^{\pi(.,\delta)}(T)-B(T,\delta)+X^{\gamma(.,\delta)}(T)+\hat{Y}(T)-\int_0^T g(s,\delta,\pi(s,\delta)+\gamma(s,\delta))ds\right)\right]. \tag{A.6}$$

Note that, the utility is same as that from policy, $(0,\gamma(t,\delta)+\pi(t,\delta))$, with no hedging. The reason is that

$$W^{\pi(.,\delta)}(T)+X^{\gamma(.,\delta)}(T)=W^{\pi(.,\delta)+\gamma(.,\delta)}(T) \tag{A.7}$$

which follows from (5). Therefore, given a linear contract, any policy involving investment in the risky stock in the manager's personal portfolio is equivalent to a policy involving no such investment. **Q.E.D.**

B. A Useful Lemma

In the proofs of our results, we will use the following lemma repeatedly.

Lemma 1 ***(Representation of Expected Utility)*** *Given $\mathcal{F}_t$-progressively measurable processes $\{\alpha(t), \beta(t)\}$ and a $\mathcal{F}_T$-measurable random variable $\xi(T)$, define certainty equivalent process $U(t)$ by*

$$-\exp\{-R_A U(t)\} = \mathbb{E}\left[-\exp\left\{-R_A\left[\xi(T)+\int_t^T \alpha(t)\,ds+\int_t^T \beta(s)\,dZ(s)\right]\right\}|\mathcal{F}_t\right]. \tag{B.1}$$

Under regularity conditions there exists a $\mathcal{F}_t$-predictable and square-integrable process $K_1(t)$such that

$$dU(t) = \left[\frac{R_A}{2}(K_1(t)+\beta(t))^2-\alpha(t)\right]dt + K_1(t)\,dZ(t), \qquad U(T)=\xi(T). \tag{B.2}$$

Integrating we have

$$U(t) = U(0)+\int_0^t \left[\frac{R_A}{2}(K_1(s)+\beta(s))^2-\alpha(s)\right]ds+\int_0^t K_1(s)\,dZ(s). \tag{B.3}$$

Proof: Note that $-\exp\left\{-R_A U(t) - R_A\int_0^t \alpha(t)\,ds - R_A\int_0^t \beta(s)\,dZ(s)\right\}$ is a martingale. Therefore, by the martingale representation theorem there exists a square-integrable process $K_2(t)$ such that

$$\frac{d\exp\left\{-R_A U(t) - R_A\int_0^t \alpha(s)\,ds - R_A\int_0^t \beta(s)\,dZ(s)\right\}}{\exp\left\{-R_A U(t) - R_A\int_0^t \alpha(t)\,ds - R_A\int_0^t \beta(s)\,dZ(s)\right\}} = -R_A K_2(t)\,dZ(t). \tag{B.4}$$

Ito's lemma then yields

$$dU(t) = -\alpha(t)\,dt + \frac{R_A}{2}(K_2(t))^2 dt + (K_2(t)-\beta(t))dZ(t). \tag{B.5}$$

Now letting $K_1(t) = K_2(t) - \beta(t)$, we have (B.3). **QED.**

C. Proof of Theorem 1

We first derive the first-best compensation *fee* and first-best investment policy which are both determined by the investor. We then derive the optimal compensation *contract*. The investor determines the contract form but does not determine (dictate) the investment policy which is determined by the manager. We then verify that the optimal contract implements the first-best investment and fee. That is, given the contract, (a) the manager optimally adopts the first-best investment policy; and (b) at the equilibrium (i.e. when the manager adopts his optimal investment), the optimal compensation contract *value* equals the first-best compensation *fee*.

C.1. Derive First-Best Equilibrium Fee and Investment Policy

Fix $\pi(t,\delta)$ and $P_F(T,\delta,S(.))$. Define the certainty equivalent process $U_F(t,\delta)$ for the manager,

$$-\exp\{-R_A U_F(t,\delta)\} = \mathbb{E}\left[-\exp\left\{-R_A\left[P_F(T,\delta,S(.)) - \int_t^T \frac{\delta}{2}\pi^2(s,\delta)\,ds\right]\right\}|\mathcal{F}_t\right]. \tag{C.1}$$

Applying (B.3), there exists a $K(t,\delta)$ such that

$$P_F(T,\delta,S(.)) = U_F(T,\delta) = U_F(0,\delta) + \int_0^T \left[\frac{R_A}{2}K^2(t,\delta) + \frac{\delta}{2}\pi^2(t,\delta)\right]dt + \int_0^T K(t,\delta)\,dZ(t), \tag{C.2}$$

where $U_F(0,\delta) = \overline{U}$ is the manager's initial utility (reservation).

Given the representation of $P_F(T,\delta,S(.))$ in (C.2) which guarantees the manager's reservation, the investor chooses the desired portfolio policy $\pi(t,\delta)$ and $K(t,\delta)$ to maximize his expected value,

$$\max_{\pi(.,\delta),K(.,\delta)} \mathbb{E}\left[W^{\pi}(T) - P_F(T,\delta,S(.))\right].$$

Using (5) and (C.2) we have[25]

$$\mathbb{E}\left[W^{\pi}(T) - P_F(T,\delta,S(.))\right] = e^{rT}W_0 - \overline{U} + \int_0^T \left[h(t)\eta\pi(t,\delta) - \frac{R_A}{2}K^2(t,\delta) - \frac{\delta}{2}\pi^2(t,\delta)\right]dt. \tag{C.3}$$

The first-order conditions yield

$$K(t,\delta) = 0, \qquad \pi_F(t,\delta) = h(t)\eta\delta^{-1}. \tag{C.4}$$

Therefore, (C.2) becomes

$$P_F(T,\delta,S(.)) = \overline{U} + \int_0^T \frac{\delta}{2}\pi_F^2(t,\delta)\,dt. \tag{C.5}$$

C.2. Derive the Optimal Compensation Contract

Define $\widetilde{U}_F(t,\delta)$ by[26]

$$-\exp\left\{-R_A\widetilde{U}_F(t,\delta)\right\} = \mathbb{E}\left[-\exp\left\{-R_A\left[\begin{array}{c} C_F(T) + e^{rT}W_0 - W(T) + \\ \int_t^T\left[h(t)\eta\,\pi(t,\delta) - \frac{\delta}{2}\pi^2(t,\delta)\right]dt - \int_t^T h(t)\sigma\pi(t,\delta)dZ(t)\end{array}\right]\right\}|\mathcal{F}_t\right]. \tag{C.6}$$

Note that $\widetilde{U}_F(T,\delta) = C_F(T) + e^{rT}W_0 - W(T)$ and $\widetilde{U}_F(0,\delta) = \overline{U}$.[27] The reason that we have defined $\widetilde{U}_F(t,\delta)$ this way is that since

$$e^{rT}W_0 - W(T) = -\int_0^T (h(s)\eta\,\pi(s,\delta)ds - \int_0^T h(s)\sigma\pi(s,\delta)dZ(s), \tag{C.7}$$

the exponent on the right hand side of (C.6) can be rewritten as

$$C_F(T) - \int_0^t (h(s)\eta\,\pi(s,\delta)ds - \int_0^t h(s)\sigma\pi(s,\delta)dZ(s) - \int_t^T \frac{\delta}{2}\pi^2(s,\delta))ds. \tag{C.8}$$

[25]It is assumed that $K(t,\delta)$ and $\pi_F(t,\delta)$ are deterministic. Their explicit solutions verify that they are. Similar assumptions are made and verified in later discussions and proofs.

[26]To simplify notation, when no confusion is likely, we omit the other arguments in $C_F(T)$.

[27]Note that $C_F(T) = W(T) - \left(e^{rT}W_0 - \widetilde{U}_F(T,\delta)\right)$ has the form of $C^a(T)$ in Proposition 1. The use of Proposition 1 is implicit but is used. Similar comment applies to later proofs.

Since $\int_0^t (h(s)\eta\,\pi(s,\delta)ds - \int_0^t h(s)\sigma\pi(s,\delta)dZ(s)$ is known at time t, maximizing the manager's utility resulting from $C_F(T) - \int_t^T \frac{\delta}{2}\pi^2(s,\delta))ds$ is equivalent to maximizing $\widetilde{U}_F(t,\delta)$. From (B.3), there exists a $\widetilde{K}(t,\delta)$ such that

$$\widetilde{U}_F(T,\delta) = \overline{U} + \int_0^T \left[\frac{R_A}{2}\left(h(s)\,\sigma\pi(s,\delta) - \widetilde{K}(s,\delta)\right)^2 - h(s)\eta\pi(s,\delta) + \frac{\delta}{2}\pi^2(s,\delta)\right] ds - \int_0^T \widetilde{K}(s,\delta)\,dZ(s). \tag{C.9}$$

Given $\widetilde{K}(t,\delta)$, the *manager* maximizes $\widetilde{U}_F(T,\delta)$ by choosing $\pi(t,\delta)$. The first-order condition is,

$$R_A\left[h(t)\,\sigma\widetilde{\pi}(t,\delta) - \widetilde{K}(t,\delta)\right]h(t)\sigma - h(t)\eta + \delta\widetilde{\pi}(t,\delta) = 0. \tag{C.10}$$

The *investor* chooses $\widetilde{K}(t,\delta)$ to maximize his utility (expected residual value),

$$\mathbb{E}[e^{rT}W_0 - \widetilde{U}_F(T,\delta)] = e^{rT}W_0 - \overline{U} - \int_0^T \left[\frac{R_A}{2}(h(t)\,\sigma\widetilde{\pi}(t,\delta) - \widetilde{K}(t,\delta))^2 - h(t)\widetilde{\pi} + \frac{\delta}{2}\widetilde{\pi}^2(t,\delta)\right] dt. \tag{C.11}$$

It is clear that the maximizing (note that (C.10) has been used)

$$\widetilde{K}(t,\delta) = h(t)\,\sigma\widetilde{\pi}(t,\delta). \tag{C.12}$$

From (C.10) and (C.12) we have

$$\widetilde{\pi}(t,\delta) = h(t)\eta\delta^{-1} = \pi_F(t,\delta), \qquad \widetilde{K}(t,\delta) = h(t)\,\sigma\pi_F(t,\delta). \tag{C.13}$$

It follows then that we can write $\widetilde{U}_F(T,\delta)$ as (with $\widetilde{K}(t,\delta)$ and $\pi(t,\delta)$ at their optimal values),

$$\widetilde{U}_F(T,\delta) = \overline{U} + \int_0^T \frac{\delta}{2}\pi_F^2(t,\delta)dt + e^{rT}W_0 - X^{\pi_F}(T), \tag{C.14}$$

where

$$X^{\pi_F}(T) = e^{rT}W_0 + \int_0^T h(t)\,\eta\,\pi_F(t,\delta)dt + \int_0^T h(t)\,\sigma\pi_F(t,\delta)dZ(t) \tag{C.15}$$

is the portfolio weath from following policy $\pi_F(t,\delta)$. We can now write $C(T)$ as

$$C_F(T) = W(T) - e^{rT}W_0 + \widetilde{U}_F(T,\delta) = W(T) - X^{\pi_F}(T) + \overline{U} + \int_0^T \frac{\delta}{2}\pi_F^2(t,\delta)dt. \tag{C.16}$$

Obviously, this contract achieves first-best optimal fee $P_F(T,\delta,S(.))$ when the manager chooses $\pi_F(t,\delta)$. Next, we verify that given (C.16), the manager optimally adopts $\pi_F(t,\delta)$.

C.3. Verify that the Compensation Contract in (C.16) Implements the First-Best Policies

Given the compensation contract in (C.16), the manager optimally chooses $\pi(t,\delta)$ to maximize

$$\max_{\pi(.,\delta)} \mathbb{E}\left[-\exp\left\{-R_A\left[W^\pi(T) - X^{\pi_F}(T) + \overline{U} + \int_0^T \frac{\delta}{2}\pi_F^2(t,\delta)\,dt - \int_0^T \frac{\delta}{2}\pi^2(t,\delta)\,dt\right]\right\}\right]. \tag{C.17}$$

Since

$$W^\pi(T) - X^{\pi_F}(T) = \int_0^T h(t)\eta\,[\pi(t,\delta) - \pi_F(t,\delta)]\,dt + \int_0^T h(t)\sigma\,[\pi(t,\delta) - \pi_F(t,\delta)]\,dZ(t), \tag{C.18}$$

the manager's maximization problem becomes

$$\max_{\pi(.,\delta)} \mathbb{E}\left[-\exp\left\{-R_A\left[\begin{array}{c} \int_0^T h(t)\eta\,[\pi(t,\delta) - \pi_F(t,\delta)]dt + \int_0^T \frac{\delta}{2}\pi_F^2(t,\delta)\,dt - \int_0^T \frac{\delta}{2}\pi^2(t,\delta)\,dt \\ -\frac{R_A}{2}\int_0^T h^2(t)\sigma^2\,[\pi(t,\delta) - \pi_F(t,\delta)]^2 dt \end{array}\right]\right\}\right].$$

The first-order condition yields

$$h(t)\eta - R_A h^2(t)\sigma^2\,[\pi(t,\delta) - \pi_F(t,\delta)] - \delta\pi(t,\delta) = 0; \qquad \pi(t,\delta) = h(t)\eta\delta^{-1} = \pi_F(t,\delta). \tag{C.19}$$

Given that the manager's optimal (equilibrium) investment policy is $\pi(t,\delta) = \pi_F(t,\delta)$, $W^\pi(T) - X^{\pi_F}(T) = 0$. The compensation contract in (C.16) reduces to the first-best fee in (C.5) in equilibrium, and for this case (with no skill uncertainty), the second-best and first-best coincide. **Q.E.D.**

D. Proof of Theorem 2

We first derive the condition on the compensation that induces the manager to report his skill truthfully. We then derive the compensation fee and second-best investment policy which are both determined by the investor.

D.1. Representation of the Truth Telling (TT) Condition

First, note that we still have the same representation of compensation fee (C.2),

$$P_S(T,\delta,S(.)) = U_S(T,\delta) = U_S(0,\delta) + \int_0^T \left[\frac{R_A}{2}K^2(t,\delta) + \frac{\delta}{2}\pi^2(t,\delta)\right] ds + \int_0^T K(t,\delta)\,dZ(t). \tag{D.1}$$

Assume that the manager with skill δ reports skill δ'. Given that the manager reports δ', the investor dictates $\pi(t,\delta')$ and pays the manager $P_S(T,\delta',S(.))$. The manager's certainty equivalent $U_S(0,\delta,\delta')$ is determined by[28]

$$-\exp\left(-R_A U_S\left(0,\delta,\delta'\right)\right) = \mathbb{E}\left[-\exp\left\{-R_A\left[U_S\left(T,\delta'\right) - \int_0^T \frac{\delta}{2}\pi^2(t,\delta')dt\right]\right\}\right]. \tag{D.2}$$

Note that $\int_0^T \frac{\delta}{2}\pi^2(t,\delta')$ is the cost to the manager with skill δ even if he reports δ'. Plugging $U_S(T,\delta')$ from (D.1) into (D.2), we have

$$-\exp\left(-R_A U_S\left(0,\delta,\delta'\right)\right) = -\exp\left\{-R_A\left[U_S\left(0,\delta'\right) + \int_0^T \frac{\delta'}{2}\pi^2(t,\delta')ds - \int_0^T \frac{\delta}{2}\pi^2(t,\delta')dt\right]\right\}\mathbb{E}\left[M\left(T,\delta'\right)\right], \tag{D.3}$$

where

$$\mathbb{E}\left[M\left(T,\delta'\right)\right] = \mathbb{E}\left[\exp\left\{-\frac{1}{2}\int_0^T R_A^2 K^2\left(t,\delta'\right)dt - \int_0^T R_A K\left(t,\delta'\right)dZ(t)\right\}\right] = 1. \tag{D.4}$$

Truth telling requires that

$$\delta \in \arg\max_{\delta'} \; U_S\left(0,\delta'\right) + \int_0^T \frac{\delta'}{2}\pi^2(t,\delta')dt - \int_0^T \frac{\delta}{2}\pi^2(t,\delta')dt = U_S(0,\delta). \tag{D.5}$$

[28] To abuse notation a bit, we use the same $U_S()$ to denote $U_S(0,\delta')$ and $U_S(0,\delta,\delta')$. Also note that we should have used $\pi(t,\delta,\delta')$ to denote the manager's trading when his true skill is δ but he reports δ'. We omit δ in $\pi(t,\delta,\delta')$ to save notations.

That is, the manager's maximum achievable initial utility is $U_S(0,\delta)$ which results from his true skill. The first-order derivative of the left hand side evaluated at $\delta' = \delta$ is zero,

$$\frac{\partial U_S(0,\delta)}{\partial \delta} + \int_0^T \frac{1}{2}\pi^2(t,\delta)dt = 0. \tag{D.6}$$

Integrating we have

$$U_S(0,\delta) = \overline{U} + \int_\delta^{\bar{\delta}} \left\{ \int_0^T \frac{1}{2}\pi^2(s,\tau)\,ds \right\} d\tau, \tag{D.7}$$

where $\overline{U} = U_S\left(0,\bar{\delta}\right)$ is the certainty-equivalent of the most inefficient agent with skill $\bar{\delta}$. Now (D.1) becomes

$$P_S(T,\delta,S(.)) = \overline{U} + \int_\delta^{\bar{\delta}} \left\{ \int_0^T \frac{1}{2}\pi^2(t,\tau)\,dt \right\} d\tau + \int_0^T \left[\frac{R_A}{2} K^2(t,\delta) + \frac{\delta}{2}\pi^2(t,\delta) \right] dt + \int_0^T K(t,\delta)\,dZ(t). \tag{D.8}$$

D.2. Derive the Investor's Optimal $\pi(t,\delta)$ and $K(t,\delta)$.

The investor chooses $\pi(t,\delta)$ and $K(t,\delta)$ to maximize

$$\int_{\underline{\delta}}^{\bar{\delta}} \mathbb{E}\left[W^{\pi(.,\delta)}(T) - P_S(T,\delta,S(.)) \right] f(\delta)\,d\delta.$$

Using (D.8) for $P_S(T,\delta,S(.))$ and (5) for $W^{\pi(.,\delta)}(T)$, we have

$$\mathbb{E}\left[W^{\pi(.,\delta)}(T) - P_S(T,\delta,S(.)) \right] = e^{rT}W_0 - \overline{U} + \int_0^T \left[h(t)\eta\,\pi(t,\delta) - \frac{R_A}{2}K^2(s,\delta) - \frac{\delta}{2}\pi^2(s,\delta) - \int_\delta^{\bar{\delta}} \frac{1}{2}\pi^2(t,\tau)\,d\tau \right] dt. \tag{D.9}$$

By integration by parts we have

$$\int_{\underline{\delta}}^{\bar{\delta}} \left(\int_\delta^{\bar{\delta}} \frac{1}{2}\pi^2(t,\tau)\,d\tau \right) f(\delta)d\delta = F(\delta) \left(\int_\delta^{\bar{\delta}} \frac{1}{2}\pi^2(t,\tau)\,d\tau \right) \Big|_{\underline{\delta}}^{\bar{\delta}} + \int_{\underline{\delta}}^{\bar{\delta}} F(\delta)\frac{1}{2}\pi^2(t,\delta)\,d\delta = \int_{\underline{\delta}}^{\bar{\delta}} \frac{F(\delta)}{2}\pi^2(t,\delta)\,d\delta. \tag{D.10}$$

Therefore, the investor's problem becomes

$$\max_{\pi(t.,\delta),K(t,\delta)} \int_{\underline{\delta}}^{\bar{\delta}} \int_0^T \left[h(t)\eta\pi(t,\delta) - \frac{R_A}{2}K^2(t,\delta) - \frac{\delta}{2}\pi^2(t,\delta) - \frac{F(\delta)}{2f(\delta)}\pi^2(t,\delta) \right] f(\delta)dt. \tag{D.11}$$

From the first-order conditions for $K(t,\delta)$ and $\pi(t,\delta)$, we have

$$K_S(t,\delta) = 0, \qquad \pi_S(t,\delta) = h(t)\left[\delta + \frac{F(\delta)}{f(\delta)}\right]^{-1}\eta. \tag{D.12}$$

Since $K(t,\delta) = 0$, the second-best fee in (D.1) becomes

$$P_S(T,\delta) = U_S(0,\delta) + \int_0^T \frac{\delta}{2}\pi_S^2(t,\delta)dt, \tag{D.13}$$

where $U_S(0,\delta)$ is given in (D.7) and can be written as $U_S(0,\delta) = \overline{U} + I\mathcal{M}_S(\delta)$ where

$$I\mathcal{M}_S(\delta) = \int_\delta^{\bar{\delta}}\int_0^T \frac{1}{2}\pi_S^2(t,\tau)dtd\tau = \frac{(e^{2rT}-1)\eta^2}{4r}\int_\delta^{\bar{\delta}}\left[\tau + \frac{F(\tau)}{f(\tau)}\right]^{-2}d\tau \tag{D.14}$$

is the informational rent which is increasing and zero for the most inefficient manager with $\delta = \bar{\delta}$. **Q.E.D.**

E. Proof of Theorem 3

E.1. Representation of the Truthful Enforcement (TE) Conditions

First, note that following similar steps in Subsection C.2, we still have $\widetilde{U}_D(T,\delta) = C_D(T) + e^{rT}W_0 - W(T)$ where $\widetilde{U}_D(T,\delta)$ is given by (C.9),

$$\widetilde{U}_D(T,\delta) = \widetilde{U}_D(0,\delta) + \int_0^T\left[\frac{R_A}{2}\left(h(t)\sigma\pi(t,\delta) - K(t,\delta)\right)^2 - h(t)\eta\pi(t,\delta) + \frac{\delta}{2}\pi^2(t,\delta)\right]dt - \int_0^T K(t)\,dZ(t). \tag{E.1}$$

The manager can misreport his skill and adopt a different policy. We now derive the conditions under which the manager truthfully reports his skill δ and adopts $\pi(t,\delta)$ which is the optimal one for skill δ. To this end, assume that the manager with skill δ reports skill δ' and chooses portfolio $\widehat{\pi}(t,\delta')$. Given that the manager reports δ', the investor pays the manager $C_D(T,\delta') = W^{\widehat{\pi}}(T) - e^{rT}W_0 + \widetilde{U}_D(T,\delta')$. The manager's initial certainty-equivalent wealth $U_D(0,\delta,\delta')$ is determined by

$$-\exp\left(-R_A U_D\left(0,\delta,\delta'\right)\right) = \mathbb{E}\left[-\exp\left\{-R_A\left[W^{\widehat{\pi}}(T) - e^{rT}W_0 + \widetilde{U}_D(T,\delta') - \int_0^T \frac{\delta}{2}\widehat{\pi}^2(t,\delta')dt\right]\right\}\right]. \tag{E.2}$$

Note that $\int_0^T \frac{\delta}{2}\widehat{\pi}^2(t,\delta')$ is the cost to the manager with skill δ even if he reports δ'. Since

$$\begin{aligned} W^{\widehat{\pi}}(T) - e^{rT}W_0 + \widetilde{U}_D(T,\delta') = \widetilde{U}_D(0,\delta') + \int_0^T \left[h(t)\eta\left(\widehat{\pi}(t,\delta') - \pi(t,\delta')\right) + \frac{R_A}{2}\left(h(t)\sigma\pi(t,\delta') - K\left(t,\delta'\right)\right)^2 + \frac{\delta'}{2}\pi^2(t,\delta')\right]dt \\ + \int_0^T \left[h(t)\sigma\widehat{\pi}(t,\delta') - K\left(t,\delta'\right)\right]dZ(t), \end{aligned} \tag{E.3}$$

we have

$$-\exp\left(-R_A U_D\left(0,\delta,\delta'\right)\right) = -\exp\left\{-R_A\left[\widetilde{U}_D\left(0,\delta'\right) + \int_0^T \left[\begin{array}{c} h(t)\eta\left(\widehat{\pi}(t,\delta') - \pi(t,\delta')\right) + \frac{R_A}{2}\left(h(t)\sigma\pi(t,\delta') - K\left(t,\delta'\right)\right)^2 + \\ \frac{\delta'}{2}\pi^2(t,\delta') - \frac{\delta}{2}\widehat{\pi}^2(t,\delta') - \frac{R_A}{2}\left(h(t)\sigma\widehat{\pi}(t,\delta') - K\left(t,\delta'\right)\right)^2 \end{array}\right]dt\right]\right\}.$$

Note that $\pi(t,\delta')$ which appears in $\widetilde{U}_D(T,\delta')$ is different from $\widehat{\pi}(t,\delta')$ and not controlled by the manager.

Truth telling *and* adopting policy under truth telling (i.e. truthful enforcement) require that

$$(\delta,\pi(t,\delta)) \in \max_{\delta',\pi(t,\delta')} \widetilde{U}_D\left(0,\delta'\right) + \int_0^T \left[\begin{array}{c} h(t)\eta\left(\widehat{\pi}(t,\delta') - \pi(t,\delta')\right) + \frac{R_A}{2}\left(h(t)\sigma\pi(t,\delta') - K\left(t,\delta'\right)\right)^2 + \\ \frac{\delta'}{2}\pi^2(t,\delta') - \frac{\delta}{2}\widehat{\pi}^2(t,\delta') - \frac{R_A}{2}\left(h(t)\sigma\widehat{\pi}(t,\delta') - K\left(t,\delta'\right)\right)^2 \end{array}\right]dt = \widetilde{U}_D\left(0,\delta\right).$$

The first-order derivatives with respect to (δ', $\widehat{\pi}(t,\delta')$) evaluted at (δ, $\pi(t,\delta)$) are set to zero,

$$\frac{\partial \widetilde{U}_D\left(0,\delta\right)}{\partial \delta} + \int_0^T \frac{1}{2}\pi^2(t,\delta)dt = 0. \tag{E.4}$$

$$h(t)\sigma\pi(t,\delta) - K\left(t,\delta\right) = \frac{h(t)\eta}{\sigma R_A} - \frac{\delta\pi(t,\delta)}{h(t)\sigma R_A}. \tag{E.5}$$

Integrating (E.4) we have

$$\widetilde{U}_D\left(0,\delta\right) = \overline{U} + \int_\delta^{\bar{\delta}} \left\{\int_0^T \frac{1}{2}\pi^2\left(s,\tau\right)ds\right\}d\tau. \tag{E.6}$$

where $\overline{U} = \widetilde{U}_D\left(0,\bar{\delta}\right)$ is the certainty-equivalent of the most inefficient agent with skill $\bar{\delta}$.

E.2. Derive the Investor's Optimal $K(t,\delta)$

The investor chooses $K(t,\delta)$ to maximize his expected value subject to (E.5) and (E.6),

$$\max_{K(t,\delta)} \int_{\delta}^{\bar{\delta}} \mathbb{E}\left[W^{\pi(\delta)}(T) - C(T,\delta)\right] f(\delta)\, d\delta = \max_{K(t,\delta)} \int_{\delta}^{\bar{\delta}} \mathbb{E}\left[e^{rT}W_0 - \widetilde{U}(T,\delta)\right] f(\delta)\, d\delta =$$
$$\max_{K(t,\delta)} -\int_0^T \left[\frac{R_A}{2}\left(h(t)\,\sigma\pi(t,\delta) - K(t,\delta)\right)^2 - h(t)\eta\pi(t,\delta) + \frac{\delta}{2}\pi^2(t,\delta) + \frac{F(\delta)}{2f(\delta)}\pi^2(t,\delta)\right] f(\delta)dt \qquad \text{(E.7)}$$

where (D.10) has been used. The first-order condition with respect to $K(t,\delta)$ is,

$$R_A\left(\frac{h(t)\eta}{\sigma R_A} - \frac{\delta\pi(t,\delta)}{h(t)\sigma R_A}\right)\left(\frac{\sigma R_A}{\delta + R_A h^2(t)\sigma^2} - 1\right) + \left(\delta\pi(t,\delta) + \frac{F(\delta)}{f(\delta)}\pi(t,\delta) - h(t)\eta\right)\frac{\sigma R_A}{\delta + R_A h^2(t)\sigma^2} = 0. \qquad \text{(E.8)}$$

Solving (E.5) and (E.8) yields

$$\pi_D(t,\delta) = h(t)\left[\delta + \frac{R_A h^2(t)\sigma^2}{\delta + R_A h^2(t)\sigma^2}\frac{F(\delta)}{f(\delta)}\right]^{-1}\eta, \qquad K_D(t,\delta) = \frac{\delta + R_A h^2(t)\sigma^2}{R_A h(t)\sigma}\pi_D(t,\delta) - \frac{\eta}{\sigma R_A}. \qquad \text{(E.9)}$$

E.3. Deriving the Compensation Contract

Define a benchmark portfolio with portfolio policy $\nu_D(t,\delta) = K_D(t,\delta)/(h(t)\sigma)$. Its wealth process follows

$$X^{\nu_D}(T,\delta) = e^{rT}W_0 + \int_0^T h(t)\eta\,\nu_D(t,\delta)dt + \int_0^T h(t)\sigma\nu_D(t,\delta)dZ(t). \qquad \text{(E.10)}$$

At equilibrium, we can rewrite (E.1) as

$$\widetilde{U}_D(T,\delta) = \widetilde{U}_D(0,\delta) + e^{rT}W_0 - X^{\nu_D}(T,\delta) + \int_0^T \left[\begin{array}{c} \frac{R_A}{2}h^2(t)\sigma^2\left(\pi_D(t,\delta) - \nu_D(t,\delta)\right)^2 - \\ h(t)\eta\left(\pi(t,\delta) - \nu_D(t,\delta)\right) + \frac{\delta}{2}\pi_D^2(t,\delta) \end{array}\right] dt \qquad \text{(E.11)}$$

where

$$\widetilde{U}_D(0,\delta) = \overline{U} + \int_{\delta}^{\bar{\delta}}\left\{\int_0^T \frac{1}{2}\pi_D^2(s,\tau)\,ds\right\} d\tau. \qquad \text{(E.12)}$$

Since

$$h(t)\,\sigma(\pi_D(t,\delta)-\nu_D(t,\delta))=\frac{\eta}{\sigma R_A}-\frac{\delta}{h(t)\,\sigma R_A}\pi_D(t,\delta)=$$
$$\frac{\delta}{h(t)\,\sigma R_A}\left(\frac{h(t)\eta}{\delta}-\pi_D(t,\delta)\right)=\frac{\delta}{h(t)\,\sigma R_A}\left(\pi_F(t,\delta)-\pi_D(t,\delta)\right), \tag{E.13}$$

we can rewrite $\widetilde{U}_D(T,\delta)$ as

$$\widetilde{U}_D(T,\delta)=\widetilde{U}_D(0,\delta)+e^{rT}W_0-X^{\nu_D}(T)+$$
$$\int_0^T\left[\frac{\delta^2}{2h^2(t)\,\sigma^2R_A}(\pi_F(t,\delta)-\pi_D(t,\delta))^2-\frac{\delta\eta}{h(t)\,\sigma^2R_A}(\pi_F(t,\delta)-\pi_D(t,\delta))+\frac{\delta}{2}\pi_D^2(t,\delta)\right]dt. \tag{E.14}$$

Now we design the implementable compensation contract (i.e. not dependent on manager's $\pi(t,\delta)$) as

$$C_D(T,\delta)=W(T)-e^{rT}W_0+\widetilde{U}_D(T,\delta)=\underbrace{W(T)-X^{\nu_D}(T)}_{excess\ return}+\underbrace{\widetilde{U}_D(0,\delta)-\mathcal{UM}_D(\delta)}_{Cash}+\underbrace{\int_0^T\frac{\delta}{2}\pi_D^2(t,\delta)\,dt}_{Payment\ of\ cost}, \tag{E.15}$$

where

$$\mathcal{UM}_D(\delta)=\int_0^T\left[\frac{\delta\eta}{h(t)\,\sigma^2R_A}(\pi_F(t,\delta)-\pi_D(t,\delta))-\frac{\delta^2}{2h^2(t)\,\sigma^2R_A}(\pi_F(t,\delta)-\pi_D(t,\delta))^2\right]dt, \tag{E.16}$$

is the under-performance measure, and $U_D(0,\delta)=\overline{U}+IM_D(\delta)$ where

$$I\mathcal{M}_D(\delta)=\int_\delta^{\bar{\delta}}\left\{\int_0^T\frac{1}{2}\pi_D^2(s,\tau)\,ds\right\}d\tau \tag{E.17}$$

is the informational rent. The equilibrium compensation fee is

$$P_D(T,\delta,S(.))=\underbrace{W^{\pi_D(.)}(T)-X^{\nu_D}(T)}_{excess\ return}+\underbrace{\widetilde{U}(0,\delta)-\mathcal{UM}_D(\delta)}_{Cash}+\underbrace{\int_0^T\frac{\delta}{2}\pi_D^2(t,\delta)\,dt.}_{Payment\ of\ cost} \tag{E.18}$$

Remark 6 *Clearly, both* $\pi_D(t,\delta)$ *and* $P_D(T,\delta,S(.))$ *are different from their counterparts in the second-best problem. It follows that the second-best fee and portfolio policy are not implementable.*

E.4. Verify That the Compensation Contract in (E.15) Implements Truth Telling and $\pi_D(t,\delta)$

Assume that the manager wih skill δ reports δ' and adopts portfolio $\widehat{\pi}(t)$. Given that the manager reports δ', the investor pays the manager $C_D(T,\delta')$. Define the manager's certainty-equivalent $\widetilde{U}_D(0,\delta,\delta')$ by

$$-\exp\left(-R_A\widetilde{U}_D\left(0,\delta,\delta'\right)\right)=\mathbb{E}\left[-\exp\left\{-R_A\left[\begin{array}{c}W^{\widehat{\pi}}(T)-X^{\nu_D}(T,\delta')+\widetilde{U}_D\left(0,\delta'\right)-\mathcal{UM}_D\left(\delta'\right)\\+\int_0^T\frac{\delta'}{2}\pi_D^2\left(t,\delta'\right)dt-\int_0^T\frac{\delta}{2}\widehat{\pi}^2(t)dt\end{array}\right]\right\}\right]. \quad \text{(E.19)}$$

Note that

$$W^{\widehat{\pi}}(T)-X^{\nu_D}(T,\delta')=\int_0^T h(t)\eta\left(\widehat{\pi}(t)-\nu_D(t,\delta')\right)dt+\int_0^T h(t)\sigma\left(\widehat{\pi}(t)-\nu_D(t,\delta')\right)dZ(t). \quad \text{(E.20)}$$

Therefore

$$-\exp\left(-R_A\widetilde{U}\left(0,\delta,\delta'\right)\right)=-\exp\left\{-R_A\left(\overline{U}+\int_0^T G(t,\delta',\widehat{\pi}(t);\delta)dt\right)\right\}, \quad \text{(E.21)}$$

where

$$G(t,\delta',\widehat{\pi},\delta)=h(t)\eta\left(\widehat{\pi}(t)-\nu_D(t,\delta')\right)-\frac{R_A}{2}h^2(t)\sigma^2\left(\widehat{\pi}(t)-\nu_D(t,\delta')\right)^2+\int_{\delta'}^{\bar{\delta}}\frac{1}{2}\pi_D^2\left(s,\tau\right)d\tau-$$
$$\frac{\delta\eta}{h\left(t\right)\sigma^2R_A}\left(\pi_D\left(t,\delta'\right)-\pi_F\left(t,\delta'\right)\right)+\frac{\delta^2}{2h^2\left(t\right)\sigma^2R_A}\left(\pi_F\left(t,\delta'\right)-\pi_D\left(t,\delta'\right)\right)^2+\frac{\delta'}{2}\pi_D^2\left(t,\delta'\right)-\frac{\delta}{2}\widehat{\pi}^2(t)T. \quad \text{(E.22)}$$

The first-order conditions for ($\widehat{\pi}(t)$, δ') are given by (after collecting and canceling terms),

$$h(t)\eta-R_Ah^2(t)\sigma^2(\widehat{\pi}(t)-\nu_D(t,\delta'))-\delta\widehat{\pi}(t)=0, \quad \text{(E.23)}$$

$$h^2(t)\sigma^2\left(\widehat{\pi}(t)-\pi_D(t,\delta')\right)\frac{\partial\nu_D(t,\delta')}{\partial\delta'}+(\delta'\pi_D(t,\delta')-h(t)\eta+R_Ah^2(t)\sigma^2(\widehat{\pi}(t)-\nu_D(t,\delta'))=0. \quad \text{(E.24)}$$

The solutions are clearly $\delta'=\delta$ and $\widehat{\pi}(t)=\pi_D(t,\delta)$. **Q.E.D.**

F. Proof of Theorem 4

The proof follows closely that of Theorem 1 in Appendix C. We supply the derivations only where they differ. The investor chooses the first-best portfolio policy $\pi(t,\delta)$ and $K(t,\delta)$ to maximize his expected utility,

$$\max_{\pi(.,\delta),K(.,\delta)} \mathbb{E}\left[-\exp\{-R_P(W^\pi(T)-P_F(T,\delta,S(.)))\}\right].$$

Using (5) and (C.2) we have

$$\begin{aligned} W^\pi(T)-P_F(T,\delta,S(.)) =& e^{rT}W_0-\overline{U}+\int_0^T\left[h(t)\eta\pi(t,\delta)-\frac{R_A}{2}K^2(t,\delta)-\frac{\delta}{2}\pi^2(t,\delta)\right]dt+ \\ & \int_0^T\left[h(t)\sigma\pi(t,\delta)-K(t,\delta)\right]dZ(t). \end{aligned} \tag{F.1}$$

The objective function is equivalent to

$$\max_{\pi(.,\delta),K(.,\delta)}\int_0^T\left[h(t)\eta\pi(t,\delta)-\frac{R_A}{2}K^2(t,\delta)-\frac{\delta}{2}\pi^2(t,\delta)\right]dt-\frac{R_P}{2}\int_0^T\left[h(t)\sigma\pi(t,\delta)-K(t,\delta)\right]^2dt.$$

It is easy to check that the optimal $\pi(t,\delta)$ and $K(t,\delta)$ are given by

$$\pi_F(t,\delta)=\frac{h(t)\eta\delta^{-1}}{1+\frac{R_AR_P}{R_A+R_P}h^2(t)\sigma^2\delta^{-1}}, \qquad K_F(t,\delta)=\frac{R_P}{R_A+R_P}h(t)\sigma\pi_F(t,\delta). \tag{F.2}$$

$P_F(T,\delta,S(.))$ is given by (C.2) with $\pi(t,\delta)$ and $K(t,\delta)$ defined in (F.2).

F.1. Derive the Optimal Compensation Contract

We still have (C.9) and (C.10). Corresponding to (C.11), the investor chooses $\widetilde{K}(t,\delta)$ to maximize the following objective function,

$$\int_0^T\left[-\frac{R_A}{2}(h(t)\sigma\pi(t,\delta)-\widetilde{K}(t,\delta))^2+h(t)\eta\pi(t,\delta)-\frac{\delta}{2}\pi^2(t,\delta)-\frac{R_P}{2}\widetilde{K}^2(t,\delta)\right]dt.$$

The first-order condition with respect to $\widetilde{K}(t,\delta)$ is,

$$R_A\left[h(t)\,\sigma\pi(t,\delta)-\widetilde{K}(t,\delta)\right]-R_P\widetilde{K}(t,\delta)=0. \tag{F.3}$$

From (C.10) and (F.3) we have

$$\pi_F(t,\delta)=\frac{h(t)\eta\delta^{-1}}{1+\frac{R_AR_P}{R_A+R_P}h^2(t)\sigma^2\delta^{-1}},\qquad \widetilde{K}_F(t,\delta)=\frac{R_A}{R_A+R_P}h(t)\sigma\pi_F(t,\delta). \tag{F.4}$$

Now we design the implementable contract $C_F(T)$ as (i.e. it does not depend on $\pi(t,\delta)$)

$$\begin{aligned}&C_F(T)=W(T)-e^{rT}W_0+\widetilde{U}(T,\delta;\pi_F,\widetilde{K}_F)=W(T)-e^{rT}W_0+\overline{U}+\\&\int_0^T\left[\frac{R_A}{2}(h(t)\sigma\pi_F(t,\delta)-\widetilde{K}_F(t,\delta))^2-h(t)\eta\pi_F(t,\delta)+\frac{\delta}{2}\pi_F^2(t,\delta)\right]dt-\int_0^T\widetilde{K}_F(t,\delta)dZ(t)=\\&W(T)-X^{\widetilde{\pi}_F}(T)+\overline{U}+\int_0^T\left[\frac{R_A}{2}h^2(t)\sigma^2(\pi_F(t,\delta)-\widetilde{\pi}_F(t,\delta))^2-h(t)\eta(\pi_F(t,\delta)-\widetilde{\pi}_F(t,\delta))+\frac{\delta}{2}\pi_F^2(t,\delta)\right]dt,\end{aligned} \tag{F.5}$$

where $\widetilde{\pi}_F(t,\delta)=\widetilde{K}_F(t,\delta)/(h(t)\sigma)=\pi_F(t,\delta)R_A/(R_A+R_P)$ is the benchmark portfolio policy and

$$X^{\widetilde{\pi}_F}(T)=e^{rT}W_0+\int_0^T h(t)\eta\widetilde{\pi}_F(t,\delta)dt+\int_0^T h(t)\eta\widetilde{\pi}_F(t,\delta)dZ(t). \tag{F.6}$$

Note that the benchmark policy $\widetilde{\pi}_F(t,\delta)$ is no longer equal to the optimal fund policy $\pi_F(t,\delta)$ if $R_P>0$. The first-best equilibrium contract value is given by (by setting $\pi(t,\delta)=\pi_F(t,\delta)$ in $W(T)$)

$$C_F(T;\pi_F)=\overline{U}+\int_0^T h(t)\sigma(\pi_F(t,\delta)-\widetilde{\pi}_F(t,\delta))dZ(t)+\int_0^T\left[\frac{R_A}{2}h^2(t)\sigma^2(\pi_F(t,\delta)-\widetilde{\pi}_F(t,\delta))^2+\frac{\delta}{2}\pi_F^2(t,\delta)\right]dt. \tag{F.7}$$

Note that

$$h(t)\sigma(\pi_F(t,\delta)-\widetilde{\pi}_F(t,\delta))=\frac{R_P}{R_A+R_P}h(t)\sigma\pi_F(t,\delta) \tag{F.8}$$

which is the $K_F(t,\delta)$ in (F.2). Thus, the contract value in (F.7) achieves first-best optimal fee $P_F(T,\delta,S(.))$ when the manager chooses $\pi_F(t,\delta)$. Next, we verify that given (F.5), the manager optimally adopts $\pi_F(t,\delta)$.

F.2. Verify That the Compensation Contract in (F.5) Implements the First-Best Policy

Given the compensation contract in (F.5), the manager optimally chooses $\pi(t,\delta)$ to maximize (dropping terms which do not depend on $\pi(t,\delta)$)

$$\max_{\pi(.)} \mathbb{E}\left[-\exp\left\{-R_A\left[W^{\pi}(T) - X^{\widetilde{\pi}_F}(T) - \int_0^T \frac{\delta}{2}\pi^2(t,\delta)\,dt\right]\right\}\right].$$

The problem is equivalent to

$$\max_{\pi(.,\delta)} \int_0^T h(t)\eta\left[\pi(t,\delta) - \widetilde{\pi}_F(t,\delta)\right]dt - \int_0^T \frac{\delta}{2}\pi^2(t,\delta)\,dt - \frac{R_A}{2}\int_0^T h^2(t)\sigma^2\left[\pi(t,\delta) - \widetilde{\pi}_F(t,\delta)\right]^2 dt.$$

The first-order condition yields

$$h(t)\eta - R_A h^2(t)\sigma^2\left[\pi(t,\delta) - \widetilde{\pi}_F(t,\delta)\right] - \delta\pi(t,\delta) = 0. \tag{F.9}$$

Then,

$$\pi(t,\delta) = \frac{h(t)\eta + R_A h^2(t)\sigma^2\widetilde{\pi}_F(t,\delta)}{\delta + R_A h^2(t)\sigma^2} = \pi_F(t,\delta). \qquad \textbf{Q.E.D.} \tag{F.10}$$

G. Proof of Theorem 5

To save notation, define

$$H_S(t,\delta,\pi,K) = h(t)\eta\pi(t,\delta) - \frac{R_A}{2}K^2(t,\delta) - \frac{\delta}{2}\pi^2(t,\delta) - \frac{R_P}{2}\left(h(t)\sigma\pi(t,\delta) - K(t,\delta)\right)^2. \tag{G.1}$$

We still have (D.1) and (D.7). Corresponding to (D.9), the investor's problem becomes

$$\max_{\pi(.),K(.)} \int_{\underline{\delta}}^{\bar{\delta}} \mathbb{E}\left[-\exp\left\{-R_P(W^{\pi}(T) - P_S(T,\delta,S(.)))\right\}\right] f(\delta) d\delta =$$
$$\max_{\pi(.),K(.)} \left[-\exp\left\{-R_P\left\{-U_S(0,\delta) + \int_0^T H_S(t,\delta,\pi(t,\delta),K(t,\delta))\,dt\right\}\right\} f(\delta)\,d\delta\right], \tag{G.2}$$

subject to (D.7),

$$U_S(0,\delta) = \overline{U} + \int_{\delta}^{\bar{\delta}} \int_0^T \frac{1}{2}\pi^2(s,\tau)\,ds d\tau. \tag{G.3}$$

Unlike the risk-neutral investor case, now $U_S(0,\delta)$ appears in the exponent. We will no longer be able to use (D.10) to rewrite the objective function (G.2) as we have done in (D.11). Instead, we will treat $U_S(0,\delta)$ as a choice variable and regard (G.3) as a constraint.

Consider Lagrange multiplier $\lambda(\delta)$. Define $\xi(\delta) = \int_{\underline{\delta}}^{\delta} f(\delta)\lambda(\delta)\,d\delta$. Note that $\xi(\underline{\delta}) = 0$ and $\xi'(\delta) = f(\delta)\lambda(\delta)$. Through integration by parts, we have

$$\int_{\underline{\delta}}^{\bar{\delta}} \lambda(\delta)\left[U_S(0,\delta) - \overline{U} - \int_{\delta}^{\bar{\delta}}\int_0^T \frac{1}{2}\pi^2(s,\tau)\,dsd\tau\right] f(\delta)\,d\delta = \int_{\underline{\delta}}^{\bar{\delta}}\left[U_S(0,\delta) - \overline{U} - \int_{\delta}^{\bar{\delta}}\int_0^T \frac{1}{2}\pi^2(s,\tau)\,dsd\tau\right] d\xi(\delta)$$
$$= \int_{\underline{\delta}}^{\bar{\delta}} \left[U_S(0,\delta) - \overline{U}\right]\xi'(\delta)\,d\delta + \int_{\underline{\delta}}^{\bar{\delta}} \xi(\delta)\,d\left[\int_{\delta}^{\bar{\delta}}\int_0^T \frac{\pi^2(s,\tau)}{2} dsd\tau\right]$$
$$= \int_{\underline{\delta}}^{\bar{\delta}} \left[\xi'(\delta)\left[U_S(0,\delta) - \overline{U}\right] - \xi(\delta)\int_0^T \frac{\pi^2(s,\tau)}{2} ds\right] d\delta. \tag{G.4}$$

Hence, the investor's problem is to choose $\{U_S(0,\delta), K(t,\delta), \pi(t,\delta)\}$ to maximize the following function,

$$-\exp\left\{-R_P\left\{-U_S(0,\delta) + \int_0^T H_S(t,\delta,\pi(t,\delta),K(t,\delta))\,dt\right\}\right\} + \frac{\xi'(\delta)}{f(\delta)} U_S(0,\delta) - \frac{\xi(\delta)}{f(\delta)} \int_0^T \frac{\pi^2(t,\delta)}{2} dt. \tag{G.5}$$

The first-order condition for $U_S(0,\delta)$ yields

$$\exp\left\{-R_P\left\{-U_S(0,\delta) + \int_0^T H_S(t,\delta,\pi(t,\delta),K(t,\delta))\,dt\right\}\right\} = \frac{\xi'(\delta)}{R_P f(\delta)}. \tag{G.6}$$

Hence,

$$U_S(0,\delta) = \frac{1}{R_P} \ln \frac{\xi'(\delta)}{R_P f(\delta)} + \int_0^T H_S(t,\delta,\pi(t,\delta),K(t,\delta))\,dt. \tag{G.7}$$

Then, the investor's objective function is reduced to (after plugging in $U_S(0,\delta)$)

$$\max_{\pi(.,\delta),K(.,\delta)} \frac{\xi'(\delta)}{f(\delta)} U(0,\delta) - \frac{\xi(\delta)}{f(\delta)} \int_0^T \frac{\pi^2(t,\delta)}{2} dt = \max_{\pi(.,\delta),K(.,\delta)} \int_0^T \left[H_S(t,\delta,\pi(t,\delta),K(t,\delta)) - \frac{\xi(\delta)}{\xi'(\delta)} \frac{\pi^2(t,\delta)}{2} \right] dt. \tag{G.8}$$

Simple calculations from the first-order conditions yield[29]

$$\pi_S(t,\delta) = \frac{h(t)\eta}{\delta + \frac{\xi(\delta)}{\xi'(\delta)} + \frac{R_A R_P}{R_A+R_P} h^2(t)\sigma^2}, \qquad K_S(t,\delta) = \frac{R_P}{R_A+R_P} h(t)\sigma\pi_S(t,\delta). \tag{G.9}$$

$\xi(\delta)$ in turn is determined by plugging (G.7) and (G.9) into (G.3), i.e. (31). **Q.E.D.**

H. Proof of Theorem 6

To save notation, define

$$H_D(t,\delta,\pi,K) = h(t)\eta\pi(t,\delta) - \frac{R_A}{2}(h(t)\sigma\pi(t,\delta) - K(t,\delta))^2 - \frac{\delta}{2}\pi^2(t,\delta) - \frac{R_P}{2}K^2(t,\delta). \tag{H.1}$$

We still have (E.1), (E.5) and (E.6). Using Lagrange multiplier and following the steps in the proof of Theorem 5, we can write the investor's problem as to choose $(U_D(0,\delta), K(t,\delta))$ to maximize

$$-\exp\left\{-R_P\left\{-U_D(0,\delta) + \int_0^T H_D(t,\delta,\pi(t,\delta),K(t,\delta))\,dt\right\}\right\} + \frac{\xi'(\delta)}{f(\delta)} U_D(0,\delta) - \frac{\xi(\delta)}{f(\delta)} \int_0^T \frac{\pi^2(t,\delta)}{2} dt, \tag{H.2}$$

[29] If we have used the Lagrange multiplier technique when the investor is risk-neutral in the proof of Theorem 2, it is easy to check that $\lambda(\delta) = 1$. Thus $\xi(\delta) = F(\delta)$ and $\xi'(\delta) = f(\delta)$. Then, (G.9) becomes (D.12) when $R_P = 0$. The original proof is more straightforward.

subject to (E.5). The first-order condition with respect to $U_D(0,\delta)$ yields

$$\exp\left\{-R_P\left\{-U_D(0,\delta)+\int_0^T H_D(t,\delta,\pi(t,\delta),K(t,\delta))\,dt\right\}\right\}=\frac{\xi'(\delta)}{R_P f(\delta)}. \tag{H.3}$$

The investor's objective function is reduced to (similar to (G.8) but with $\pi(t,\delta)$ being determined by the manager)

$$\max_{K(t,\delta)}\int_0^T\left[H_D(t,\delta,\pi(t,\delta),K(t,\delta))-\frac{\xi(\delta)}{\xi'(\delta)}\frac{\pi^2(t,\delta)}{2}\right]dt. \tag{H.4}$$

The first-order condition for $K(t,\delta)$ yields

$$R_A[h(t)\sigma\pi(t,\delta)-K(t,\delta)]-R_PK(t,\delta)-\frac{\xi(\delta)}{\xi'(\delta)}\delta\pi(t,\delta)\frac{\sigma R_A}{\delta+R_Ah^2t)\sigma^2}=0 \tag{H.5}$$

where (E.5) has been used and from (E.5)

$$\frac{\partial\pi(t,\delta)}{\partial K(t,\delta)}=\frac{\sigma R_A}{\delta+R_Ah^2t)\sigma^2}. \tag{H.6}$$

Solving (E.5) and (H.5) yields

$$\pi_D(t,\delta)=h(t)\left[\delta+\frac{\xi(\delta)}{\xi'(\delta)}\frac{R_Ah^2(t)\sigma^2}{\delta+R_Ah^2(t)\sigma^2}+\frac{}{R_AR_P}R_A+R_Ph^2(t)\sigma^2\right]^{-1}\eta, \tag{H.7}$$

$$K_D(t,\delta)=\frac{\delta+R_Ah^2(t)\sigma^2}{R_Ah(t)\sigma}\pi_D(t,\delta)-\frac{\eta}{\sigma R_A}. \tag{H.8}$$

$\xi(\delta)$ in turn is determined by (38).

The implementable contract is still given by (E.15). The proof that it implements the truth telling and induces the manager to optimally choose $\pi_D(t,\delta)$ (i.e. truthful enforcement) follows the proof in Section E.4.

References

Admati, A., and P. Pfleiderer, 1997, "Does It All Add Up? Benchmarks and the Compensation of Active Portfolio Managers," *Journal of Business* 70, 323-350.

Aggarwal, R., and P. Jorion, 2010, "The Performance of Emerging Hedge Funds and Managers," *Journal of Financial Economics* 96, 238-256.

Avramov, D. and R. Wermers, 2006, "Investing in Mutual Funds When Returns Are Predicable," *Journal of Financial Economics* 81, 339-377.

Baik, M., J. Kang, and J. Kim, 2010, "Local Institutional Investors, Information Asymmetries, and Equity Returns," *Journal of Financial Economics* 97, 81-106.

Baks, K., A. Metrick, and J. Wachter, 2001, "Should Investors Avoid All Actively Managed Mutual Funds," *Journal of Financ*e 56, 45-85.

Barras, L., O. Scaillet, and R. Wermers, 2010, "False Discoveries in Mutual Fund Performance: Measuring Luck in Estimated Alphas," *Journal of Financial Economics* 97, 81-106.

Cuoco, D., and R. Kaniel (2011), "Equilibrium Prices in the Presence of Delegated Portfolio Management," *Journal of Financial Economics* 101, 264-296.

Dybvig, P., H. Farnsworth, and J. Carpenter, 2010, "Portfolio Performance and Agency," *Review of Financial Studies* 23, 1-23.

Edman, A., and X. Gabaix, 2011, "Tractability in Incentive Contracting," *Review of Financial Studies* 24, 2865-2894.

Elton, E., M. Gruber, M., S. Das, and M. Hlavka, 1993, "Efficiency with Costly Information: A Reinterpretation of Evidence from Management Portfolios," *Review of Financial Studies* 6, 1-12.

Fama, E., and K. French, 2010, "Luck versus Skill in the Cross-Section of Mutual Fund Returns," *Journal of Finance* 65, 1915- 1947.

Grinblatt, M., and Titman, S. (1989) “Adverse Risk Incentives and the Design of Performance-Based Contracts,” *Management Science*, Vol. 35, 807-822.

Grinblatt, M., and S. Titman, 1993, “Performance Measurement without Benchmarks: An Examination of Mutual Fund Returns,” *Journal of Business* 66, 47-68.

He, Z., 2009, “Optimal executive compensation when firm size follows geometric Brownian motion,” *Review of Financial Studies* 22, 859-892.

He, Z., 2011, “A model of dynamic compensation and capital structure,” *Journal of Financial Economics* 100, 351366

He, Z. and W. Xiong, 2010, “Delegated Asset Management and Investment Mandates,” working paper, University of Chicago and Princeton University.

Holmstrom, B., and P. Milgrom, 1987, “Aggregation and Linearity in the Provision of Intertemporal Incentives,” *Econometrica* 55 303-328.

Jones, C., and M. Lipson, 2004, “Are Retail Orders Different?” Working paper.

Ju, N., and X. Wan, 2011, “Optimal Compensation and Pay-Performance Sensitivity in a Continuous-Time Principal-Agent Model,” *Management Science*, forthcoming.

Kacperczyk, M., C. Sialm, and L. Zheng, 2008, “Unobserved Actions of Mutual Funds,” *Review of Financial Studies* 21, 2379-2416.

Kosowski, R., A. Timmermann, R. Wermers, and H. White, 2006, “Can Mutual Fund ”Stars” Really Pick Stocks? New Evidence From a Boootstrap Analysis,” *Journal of Finance* 61, 2551-2595.

Kyle, P., H. Ou-Yang, B. Wei, 2010, “A Model of Portfolio Delegation and Strategic Trading,” *Review of Financial Studies*, forthcoming.

Li, W., and A. Tiwari, 2009, “Incentive Contracts in Delegated Portfolio Management,” *Review of Financial Studies* 22, 4681-4714.

Ou-Yang, H., 2003, “Optimal Contracts in a Continuous-time Delegated Portfolio Management Problem” *Review of Financial Studies* 16, 172-208.

Sannikov, Y., 2008, "A continuous-time version of the principal-agent problem," *Review of Economic Studies* 75, 957-984.

Schättler, H., and J. Sung, 1993, "The First-Order Approach to Continuous-Time Principal-Agent Problem with Exponential Utility," *Journal of Economic Theory* 61, 331-371.

Stoughton, N., 1993, "Moral Hazard and the Portfolio Management Problem," *Journal of Finance* 48, 2009-2028.

Sung, J., 2005, "Optimal Contracts under Adverse Selection and Moral Hazard: A Continuous-Time Approach," *Review of Financial Studies* 18, 1021-1073.

Wermers, R., 2000, "Mutual Fund Performance: An Empirical Decomposition into Stock-Picking Talent, Style, Transactions Costs, and Expenses," *Journal of Finance* 4, 1655-1685.