Continuum Structural Optimization with Level Set Model and X-FEM*

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1. Abstract
In this paper, we implement the extended finite element method (X-FEM) combined with the level set method to solve structural shape and topology optimization problems. Numerical comparisons with the conventional finite element method in a fixed grid show that the X-FEM leads to more accurate results without increasing the mesh density and the degrees of freedom. Furthermore, the mesh in X-FEM is independent of the physical boundary of the design, so there is no need for remeshing during the optimization process. Numerical examples of mean compliance minimization in 2D are studied. The results suggest that combining the X-FEM for structural analysis with the level set based boundary representation is a promising approach to continuum structural optimization.

2. Keywords: structural optimization, shape optimization, topology optimization, extended finite element method, level set method.

3. Introduction
Engineers have followed continuum structural optimization methods with great interest in recent decades, because they can help engineers achieve better designs and save time and money. In recent decades, engineers have developed and studied many for structured optimization. In 1981 Cheng and Olhoff pioneered one of the most important original works on continuum structural optimization problems: the investigation of the optimal design of solid elastic plates [1]. This has led to a succession research work in this area. In 1988, Bendsoe and Kikuchi introduced the homogenization method [2], which later became a popular approach in topology optimization design problems. In order to improve the performance and efficiency of the homogenization method, engineers and researchers have investigated numerous variations. The most famous one is the “Solid Isotropic Microstructure with Penalization” (SIMP) method [3], developed by Zhou and Rozvany. An important reason for the success of these methods is the maturity of the finite element method which can be closely combined with the optimization models. A good example is the success of the commercial engineering optimization software OptiStruct of Altair Inc.

Given the significance of the shape representation in structure shape and topology optimization problems, the level set method was implemented in the structure optimization problems [4,5]. The level set method was first proposed by Osher and Sethian [6] and it has become very popular in many areas, such as the image processing, computer graphics, computational fluid dynamics and structure optimization. The principle of this level set framework is that the finite element analysis gives the shape and topology gradients with which the level set method can be implemented to control the evolution of the shape and topology of the design to achieve the optimal solution. However, in the optimization process, the boundary can not be always consistent with the fixed finite element mesh. If the boundary crosses over some elements, the finite element analysis around the boundary must be implemented very carefully; otherwise the discontinuity can reduce the accuracy of the result greatly. One earlier investigation is to remesh the whole design domain or a narrow band with a certain width around the boundary every iteration step to make the mesh conform to the boundary geometry. This approach results in a satisfied accuracy but becomes computational intensive as the problem scale increases. Some other approaches are developed to overcome this difficulty. A density based FEM, known as the “ersatz material” approach [4], has been widely accepted and implemented. In the density based method, weak material is adopted and the element stiffness is assumed proportional to the area fraction of the solid material within the element. This approach does not need the time-consuming remeshing process, and it is easy to operate [7]. But in the density based approach, it is very difficult to obtain accurate strain solutions near the boundary.

In our implementation, another approach, the extended finite element method (X-FEM), is applied to model arbitrarily evolving geometry without extending the scale of the FE system and, at the same time, considerably accurate results around the boundary can be achieved. The X-FEM originates from the partition of unity method

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(PUM) proposed by Melenk and Babuška [8] and the minimal remeshing method proposed by Belytschko and Black [9]. Since the minimal remeshing process which is involved in the PUM is not easy to implement for long cracks or in three dimensions, Moes et al. [10] improved this method by incorporating a discontinuous field which allows the crack interface to be totally independent of the mesh, therefore the remeshing process can be avoided completely. In 2000, the terminology X-FEM (eXtended Finite Element Method) was first proposed by Daux et al. [11] and it becomes an effective numerical approach for solving the discontinuous problems in mechanics. The basic idea of X-FEM is to augment the nodal shape function with an enrichment function on the elements along the boundary to model the arbitrarily evolving geometric features while on the other elements no further manipulation is needed. This method has been favorably implemented in simulating the crack growth and thus the remeshing process is not needed any more. The combination of the X-FEM and level set method gives an elegant way for solving structure shape and topology optimization problems. Duysinx et al. [12] and Miegroet et al. [13] implemented the X-FEM method for solving 2D and 3D parameterized shape optimization problems with semi-analytical sensitivity analysis approach and the level set description is used to represent the structure boundary. Miegroet and Duysinx [14] studied the stress concentration minimization problem using X-FEM and level set description. Edwards et al. [15] implemented the X-FEM with Evolutionary Structural Optimization (ESO), and the control points of B-spline are defined as the design variables to handle the boundary moving. In all these approaches, level set method is involved only for describing the boundary, and the evolution of the boundary is still explicitly represented, therefore the topological change of the design must be handled with additional efforts [15]. Lee et al. [16] combined the X-FEM with a nodal-based level set method. In this method, the level set contour of a density distribution is picked up as the design boundary, but this boundary is only an approximation one before convergence.

In this paper, we implement the X-FEM and level set method for solving the structure shape and topology optimization problems. The mean compliance minimization is taken as an example. The rest of this paper is organized as follows: In Section 4, we present the construction of moving implicit boundary structural optimization model with level set model. In Section 5, we introduce the X-FEM and we discuss some details and comparisons about the implementation and numerical properties. In Section 6, numerical examples are presented with a comparative study in 2D structural framework. Conclusions are finally given in Section 7.

4. Structural Optimization Problem with Level Set Model

The level set method is a high effective method developed by Osher and Sethian. In the level set framework, the design boundary $\partial \Omega$ is embedded implicitly as the zero level set of a one-dimensional-higher level set function $\Phi(x)$, which is Lipschitz-continuous, for example a 2 or 3 dimensional boundary $\{x \in \mathbb{R}^d : \Phi(x) = 0, d = 2, 3\}$. In shape and topology optimization of a solid, each part in the design domain can be defined as shown in Figure 1:

$$
\begin{align*}
\Phi(x) &= 0 & \forall x \in \partial \Omega \cap D \\
\Phi(x) &< 0 & \forall x \in D \setminus \Omega \\
\Phi(x) &> 0 & \forall x \in \Omega \setminus \partial \Omega
\end{align*}
$$

(1)

where $D \subset \mathbb{R}^d$ is a fixed design domain in which all admissible shapes $\Omega = \{x \mid x \in D, \Phi(x) \geq 0\}$ are included.

Figure 1: Definition of a 2D design domain with level set method

The implicit function $\Phi(x)$ is used to represent the boundary and optimize it, as it was originally developed for curve and surface evolution [6]. The change of the implicit function $\Phi(x)$ is governed by the Hamilton-Jacobi equation

$$
\frac{\partial \Phi(x)}{\partial t} + v_n(x)\nabla \Phi(x) = 0
$$

(2)

$v_n(x)$ is the normal velocity related to the sensitivity of the objective function with respect to the boundary variation. Hence, moving the boundary ($\Phi(x)=0$) along the normal direction is equivalent to update the level set function $\Phi$ by solving the Hamilton-Jacobi equation (Eq.(2)).

In this paper we use a linear elastic structure to describe the problem of structural optimization. With this implicit
representation and following Allaire et al. [4] and Wang et al. [5], the structural topology optimization problem can be expressed as:

\[
\begin{align*}
\text{Minimize:} & \quad J(u, \Phi) = \int_\Omega \frac{1}{2} C_{ijkl} \varepsilon_{ij}(u) \varepsilon_{kl}(u) \, d\Omega \\
\text{subject to:} & \quad \int_\Omega E_{ijkl} \varepsilon_{ij}(u) \varepsilon_{kl}(u) \, d\Omega = \int_\Omega \rho \, d\Omega + \int_{\Gamma} \tau \cdot n \, d\Gamma \\
& \quad u_{\Gamma_n} = u_0 \quad \forall \nu \in U \\
& \quad V \leq V_{\text{adm}}
\end{align*}
\]

where \( \Gamma_n \) specifies where Dirichlet boundary condition \( u = u_0 \) is defined and on \( \Gamma_t \) the boundary traction \( \tau \) is applied. \( u \) is the displacement field, \( \alpha(u) \) the strain field. The volume \( V \) of the admissible design can be written as:

\[
V = \int_\Omega d\Omega.
\]

In our approach, we use a gradient method for the minimization of the objective function \( J(\Phi) \). Given a local perturbation of the boundary of the admissible domain \( \Omega \), the resulting change in the objective function \( J(\Phi) \) according to the shape derivative [4, 5] is written as:

\[
\frac{dJ(u, \Phi)}{dt} = \int_\Gamma \beta(u, \Phi) v_n \, d\Gamma
\]

where

\[
\beta(u, \Phi) = -\frac{1}{2} C_{ijkl} \varepsilon_{ij}(u) \varepsilon_{kl}(u) + pu - \nabla \cdot (\mathbf{m} \cdot n + \nabla \cdot n(\mathbf{n}))
\]

is the shape gradient density, \( t \) is the artificial time and \( v_n \) the normal velocity representing the perturbation normal to the boundary. In the conventional discrete level set based topology optimization, a general analytical function for \( \Phi(t, x) \) is unknown. The numerical procedure for solving the H-J equation requires satisfying the CFL condition and choosing carefully the upwind schemes, extension velocities and reinitialization algorithms which severely limit the utility of the level set method. Therefore, a parameterized level set method with radial basis functions (RBFs) is developed for structural topology optimization. Readers can get more details about this method in reference [4,5,17].

5. The Extended Finite Element Method

One of the key points in the sensitivity analysis in this RBFs based level set framework is the calculation of \( \beta \) which is given in Eq.(6). Here, the promising tool X-FEM is chosen as the analysis technique. In this section, some application details and numerical properties of X-FEM are discussed.

Consider a domain \( \Omega \subset \mathbb{R}^d (d = 2 \text{ or } 3) \) bounded by boundary \( \Gamma = \Gamma_i \cup \Gamma_u \) with tractions boundary condition on \( \Gamma_t \) and displacements boundary condition on \( \Gamma_u \), the equilibrium equation and boundary conditions in a linear elasticity system can be described in strong form as shown in Eq. 1. In this paper, we consider only the small strains and displacements problems, as well as a linear material law. The weak form of the equilibrium equation is

\[
\int_\Omega \sigma(u) \cdot \varepsilon(v) \, d\Omega = \int_\Omega f \cdot v \, d\Omega + \int_{\Gamma_t} g \cdot v \, d\Gamma
\]

where \( v \) is the kinematically admissible test function.

In conventional finite element method, the displacement field is represented by

\[
u(x) = \sum_{i=1}^{n} N_i(x) u_i
\]

where \( I \) denotes index of the nodes, \( u_i \) the nodal displacements and \( N_i(x) \) the shape functions. The standard linear finite element equations can be obtained from the variational method by substituting the above equation into the weak form Eq.(7),

\[
K U = F
\]

where \( K \) is the symmetric stiffness matrix, \( F \) the load vector and \( U \) the unknown nodal displacement vector. For more details about the conventional finite element method readers can refer to Zienkiewicz et al. [18].

5.1. X-FEM for Structural Analysis

The conventional finite element method is troublesome when handling the moving discontinuities model analysis, because the mesh needs to match the geometry of the discontinuity. The X-FEM extends the conventional finite element method shape function with an augmented part which can represent particular discontinuity or singularity. Hence the discontinuous geometry boundary can be naturally approximated by the extended shape function and the time consuming remeshing process is avoided [8,9,10]. Several displacement approximation schemes have been proposed for different cases. Here we give a brief introduction about these schemes.

In X-FEM, the displacement approximation is extended to the following form:
\[ u(x) = \sum_{i \in \Omega} N_i(x)u_i + \sum_{j \in J} a_j N_j(x)\psi(x) \]  

(10)

In the above equation, the first item of the right hand side is the same as the part in Eq.(8). \( J \) denotes the index of nodes whose supports intersect the holes and inclusions interfaces. \( a_j \) are the enriched degrees of freedom on such nodes, and \( \psi(x) \) is the enrichment function. In [19], the enrichment function \( \psi(x) \) was chosen as

\[ \psi(x) = \left| \sum_{i \in \Omega} \phi_i(x)N_i(x) \right| \]  

(11)

and Moës et al.[20] used another form as follows

\[ \psi(x) = \sum_{i \in \Omega} \phi_i(x)N_i(x) \left| \sum_{i \in \Omega} \phi_i(x)N_i(x) \right| \]  

(12)

Here the level set representation is involved to approximate the location of the material interface. \( \phi_i(x) \) are the level set function values at the location \( x \).

Other than these two approaches, some enrichment functions for solving different problems have been proposed and discussed in recent literature, such as in [10,21] for crack growth, in [22] for highly localized strains in narrow damage process zones, in [23] for two-phase fluids problem, in [24] for fluid-structure interaction problem and so on. All of these approaches can be applied in the X-FEM framework. The same ideas for choosing enrichment functions can also be applied in level set method based structural optimization.

In this paper, the X-FEM scheme used originates from the modeling voids scheme [11,19]. As suggested in [11], the displacement field is approximated by

\[ u(x) = \sum_{i \in \Omega} N_i(x)H_{\Omega}(x)u_i \]  

(13)

where

\[ H_{\Omega}(x) = \begin{cases} 
1 & \text{if } x \in \Omega \\
0 & \text{if } x \not\in \Omega 
\end{cases} \]  

(14)

This X-FEM scheme was also implemented in [12,13,14] in modeling material-void interface. In practice, the displacement function is not implemented faithfully as described in Eq.(13). An understandable reason is that the displacement function in Eq.(13) is a discontinuous function in the elements cut by the boundary (Figure. 2), thus it is impossible to represent the actual displacement field and especially the strain field without augmented degree of freedom. Therefore, they implemented the displacement function just the same as Eq.(8) and merely removed the integral on the void part in calculating the element stiffness matrix. The scheme implemented in this way [11] is now different from it first proposed in [19]. This approach can be regarded as a variation of the original X-FEM for solving a set of different problems such as modeling material-void interfaces mentioned in [11]. Obviously, in practical implementation, this approach is very appealing because there is no augmented DOF involved, and the remeshing or the moving mesh algorithms are also not needed during the optimization process. Then the most important question becomes how accurate it is. This issue will be discussed in detail in this paper later.

Figure 2: Shape functions in X-FEM for a 3-node triangle element cut by the boundary

5.2. Numerical Integration Scheme

We make a modification to the integral process in our implementation because we involve the weak material (Figure 3). We do not remove completely the integral in the void part but replace the void by a very weak material to avoid the singularity in finite element analysis. This scheme is known as the “ersatz material” approach and it can be rigorously justified in some cases [4].

Figure 4 illustrate the basic ideas of the density method and X-FEM for 2D quadrilateral elements in which the
calculation of the element stiffness matrix is a major issue. When the design boundary cuts across the element edge, in density method, the stiffness matrix of the “half element” is equivalent to an intermediate material element with a density as the area ratio of the solid part to the whole element, while in X-FEM, the solid part of the element is partitioned into several parts and the element stiffness matrix is the sum of the solid material parts and the weak material part. According to the linear additive relationship of each part, we use the superposition principle to calculate the element stiffness matrix of the cut element.

The element stiffness matrix $K_E$ in X-FEM can be calculated with the following equation:

$$K_E = K_s + K_w$$

(15)

where

$$K_s = \int_{\Omega_s} B^T D_s B t H(x) \Omega_s(x) dx$$

(16)

$$K_w = \int_{\Omega_w} B^T D_w B t H(x) \Omega_w(x) dx$$

(17)

and matrix $B$ is the displacement differentiation matrix for the whole element, $t$ is the thickness and matrix $D_s$ and $D_w$ are the elasticity matrices for the solid material and weak material parts respectively. $\Omega_s$ is the integration domain in the element. $\Omega_s$ and $\Omega_w$ denote the solid material part and weak material part as illustrated in Figure 4. $K_s$ and $K_w$ are the contributions to the whole element stiffness matrix by the solid material and weak material parts respectively. Usually in $\Omega_w$, $D_w$ is a very small value compared with $D_s$ to avoid singularity and $K_w$ is also very small compared with $K_s$.

![Figure 3: The solid material and weak material](image_url)

![Figure 4: The density based method and X-FEM](image_url)

![Figure 5: Cases of solid part partition in one element of X-FEM](image_url)
Figure 5 illustrates all possible partition cases of 4 nodes quadrilateral element in our implementation. In all cases the solid parts are partitioned into sub-triangles and Gauss quadrature is used to calculate the stiffness matrix. The dash lines are the edges of the partitioned triangles for integration. It should be noticed that these triangles are only used to evaluate the integral in each element and the scale of the FE system is not increased.

To help readers to understand the difference of the integration schemes between density method and X-FEM, we consider the element stiffness matrix

\[ K_s = \int_{\Omega} B^T D_b B_t dx \]  

which is an 8-by-8 matrix in 2D 4-node quadrilateral element. Here we denote the first element of the stiffness matrix as \( K[1,1] \) and denote the first element of the integrand \( B^T D_b B_t \) as \( k[1,1] \). In Figure 6 we plot the distribution of \( k[1,1] \) in an element calculated by density method and X-FEM. Figure 6 (a) shows the distribution of \( k[1,1] \) in a normal 2D 4-node quadrilateral element with solid material; (b) shows the distribution of \( k[1,1] \) with density method and the density of the element is 50%; (c) and (d) shows \( k[1,1] \) calculated in X-FEM model with an element cut by a boundary. Obviously, in density method, we get the same result for the elements in (b), (c) and (d) because in this method we can not take into account the material distribution. The different results in (c) and (d) indicate in X-FEM the material distribution has been considered and has great influence on calculating the element stiffness matrix. Subsequent test examples illustrates that X-FEM can improve the accuracy greatly.

Figure 6: Integration schemes in density method and X-FEM

However, this scheme works well only if the function integrand \( k(x) = B^T D_b B_t \) in Eq.(18) is not uniform in the element. For example, in a 3-node triangle element shown in Figure 2, \( k(x) \) is a constant in the element because it is a constant strain element, then we have

\[ K_s = \int_{\Omega} B^T D_b B_t H_{f_c}(x)dx = B^T D_b B_t \int_{\Omega} H_{f_c}(x)dx = K_s \cdot \text{Density} \]  

Here we can see X-FEM scheme leads to the same result with density method in calculating the element stiffness matrix. This suggests that this X-FEM scheme can only prove advantages in the element types with 2nd or higher order shape functions, otherwise it has the same effect as density method.

5.3. Evaluation of the X-FEM Scheme

A 60x30 structured mesh is used in this test case. The element type is 4-node quadrilateral bi-linear element. The void part is filled with weak material with density \( \rho_0=10^{-4} \) and it is also meshed by the same type of element. We use the same type of element, the 4-node quadrilateral bi-linear elements, in the Ansys models for all the test cases in this paper. Then, the error related to the element type can be viewed as in same level.

In this example, we examine both the X-FEM implementation scheme without enriched DOF which is given by Eq. 19 and X-FEM with enriched DOFs which is given by Eq. 18. We represent the former in following figures of the test example with “X-FEM” and the latter one with “X-FEM Enr” to discriminate them.

In structural optimization, especially the mean compliance problems, the strain energy density on the boundary usually plays an important role (Eq. 9), which is calculated by
\[ E = \frac{1}{2} C_{\mu\nu} \varepsilon_{\mu\nu}(u) \varepsilon_{\mu\nu}(u) \] (20)

A set of 90 test points are chosen uniformly distributed along the round part of the boundary. The mesh and locations of the test points in Ansys are shown in Figure 8. In the Ansys analysis model, the total number of elements is 4946. In this model we refine the mesh on this part to make the nodes located in accordance with the
test points. The results with X-FEM and the density method are plotted in Figure 9, in which the red arrows denote the result of X-FEM we studied in this paper which has no enriched degree of freedom involved, the blue ones denote the result of density method, and the black ones denote the result of X-FEM with enriched degrees of freedom. A detailed comparison with Ansys is given in Figure 10, from which we can see the results of X-FEM are much more accurate than the results of density method.

![Figure 10](image)

Figure 10: Comparison of the strain energy density along the boundary with X-FEM without enriched DOF, X-FEM with enriched DOFs, density method and Ansys. Mesh size: 80×40

This result is on account of the closeness of the representation of X-FEM to the natural physical problems compared with the density based method. In density based method, people do not take account of the variation of locations of the material distribution in an element in calculating the element stiffness matrix. Material is considered uniformly distributed throughout the whole element no matter where the boundary cut through the element. In X-FEM, material distribution is taken into account by means of the integral domain in the element which is cut by the boundary as shown in Figure 4. The model of X-FEM gives a more reasonable description of the real problem and this provides a satisfactory explanation why X-FEM gives much more accurate result compared to the density based method. In addition, it should be noticed that we do not apply any stress smoothness scheme in our implementation.

The results of X-FEM scheme with and without enriched DOF show no much difference. This means the enrichment function given in Eq.(13) is not a good choice to improve the accuracy in this example. We still need further study to find more appropriate one to obtain better results. Here we prefer to implement the X-FEM without enriched DOF in the structural optimization problems because of its fixed DOFs, simplified implementation and greatly improved accuracy. Therefore, in the following numerical example of this paper, we only implement the X-FEM scheme without enriched DOF in structural optimization problem in the level set model.

### 6. Numerical Examples of Structural Optimization

In this section, we implement the X-FEM scheme into the structural shape and topology optimization problem. A minimum mean compliance problem of a short cantilever beam shown in Figure 11 is solved with parametric RBFs based level set method. The optimality criteria method is applied as the optimization algorithm. The basis parameters are assumed to be the Young’s modulus \( E = 1 \) for solid material, \( E_0 = 10^{-3} \) for weak material, Poisson’s ratio \( \nu = 0.3 \), the structure dimensions \( H = 10, L = 20 \), thickness \( t = 1.0 \), load \( P = 1 \) and limited volume fraction up to 50% is applied with the bisection method or augmented Lagrange method. In these examples, we use Wendland C2-CSRBF and the knot points are distributed uniformly in the design domain and coincide fixedly with the finite element nodes.

The design domain of this problem consists of 80×40 2D 4-node quadrilateral elements and the red ones are partitioned triangles for X-FEM integration as shown in Figure 12. The optimized final design after a fixed iteration number 60 is obtained and the volume fraction is 50%.

Here the optimality criteria method is implemented as the optimization algorithm and bisection method is used to control the volume constraint. The small blue arrows in Figure 12 indicate the strain energy density on the boundary in the normal direction. The length of an arrow is proportional to the magnitude of the energy density value on that point. We can find that in the optimized structure, these arrows almost have a same size in length which means that the strain energy densities along the boundary are distributed uniformly. This result follows what Pedersen [25] has proved that the stiffest design will have a uniform energy density along the design boundary.

The comparison of the results obtained with a coarse mesh size (40×20) is given in Figure 13. The results of the density method and X-FEM are generated from a same set of parameters and iteration steps. The result obtained
with X-FEM based method can be verified and compared with previous literature [17] which was implemented with better meshes. Though the results of these two results shown in Figure 13 are similar, the X-FEM based approach leads to more smooth boundaries. In both approaches, no additional boundary smoothness scheme is applied.

Figure 11: The definition of the minimum compliance design problem of a short cantilever beam and final design

Figure 12: Mostly uniformly distributed strain energy density along the optimized design boundary.

Figure 13: Comparison of the final design with a coarse ground mesh. Mesh size: 40×20

7. Conclusion
In the present study, we implement the effective tool X-FEM and the RBFs based level set method for structural shape and topology optimization. Other than the conventional FE model, in our study a readily-implement superposition scheme which combines the ersatz material model with X-FEM is implemented, with which a set of background integration triangle partitions are generated to match dynamic implicit boundary during the optimization process and thus the moving mesh or remeshing process are avoided. Furthermore, the test case shows that X-FEM produces more accurate results on the boundary than density method and this superiority enables the coarse mesh to be used in solving the structural optimization problems to save the computational cost. The numerical example of topology design shows the superiority of X-FEM in the accuracy, efficiency and the representation fidelity. We suggest that combining X-FEM with level set method is a promising tool for solving structural shape and topology optimization problems with implicit boundary representation.

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9. References