A level set based method for topology optimization of continuum structures with stress constraint

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Abstract
This paper presents a stress constrained topology optimization problem and its level set based solution. The volume of material is minimized, and stress constraints at all the points in a structure are efficiently aggregated into a single equivalent global constraint. Using the material derivative and adjoint method we derived the shape sensitivity. The steepest descent optimization algorithm is implemented. Several numerical examples in two dimensions are provided and discussed.

Keywords: stress constraint, constraint aggregation, topology optimization, level set, continuum structure

1. Introduction
Over the past decades topology optimization of continuum structures has become an important research topic, and its design criteria have been extended from compliance to other ones [1]. However, most of the existing design criteria consider global behavior of continuum structures, while criteria concerning local behavior catch relative little attention.

Strength criterion, considering local stress and characterizing a structure’s resistance to failure (e.g., crack), is an important design criterion in engineering applications. In the literature of topology optimization of continuum structures there exist two approaches to deal with the stress.

The first approach deals with the stress in the objective function of an optimization problem and treats the volume of material as constraint. A weighted combination of compliance and p-norm of stress was used as an objective function [2]. The maximum stress of a continuum structure was minimized via sequential element elimination and addition [3]. A domain integral of stress was treated as an objective function and minimized via the level set based method [4].

The second approach deals with the stress as constraint of an optimization problem and treats the volume of material as objective function. A topology optimization problem that takes local stresses in a continuum structure as constraints was defined by Duysinx and Bendsoe and was solved using the SIMP (Solid Isotropic Microstructure with Penalization) method [5]. In contrast to the first approach, the second approach is able to strictly enforce stress constraints and much effort was thereafter made on it [6-8].

Despite the different approaches used to treat the stress, many of the above-mentioned studies [5-8] employ the well-known SIMP method to solve the topology optimization of continuum structures. SIMP method uses continuous variables (density of material distribution [9,10]) to relax the topology optimization problem that is essentially a 0-1 optimization problem to a much easier continuous variable optimization problem. Usually, penalty is applied on the continuous variable to drive the design to “black and white” patterns to approximate the 0-1 nature of topology optimization [9,10].

When the SIMP method is used for the stress constrained topology optimization of continuum structures, two issues need to be addressed. Firstly, the intermediate densities arising during optimization represent material microstructures [11] that lead to concentration and fluctuation of microscale stress and make the stress criterion of microscale be different from that of macroscale [5,12]. Duysinx and Bendsoe obtained a model of the stress criterion of microscale for the simple isotropic microstructures of the SIMP method [5]. Secondly, the stress constraint presents the so-called “singularity phenomenon” [13] which means that the optimal solution is a singular point in the design space, and that standard optimization algorithms are unable to remove some low-density regions. The reason of such singularity phenomenon lies in the discontinuity of stress constraint when the density approaches zero [14]. In order to deal with the singularity phenomenon, $\epsilon$-relaxation method [15] or its variants [16] are often used.

Other methods are also employed for solving the stress-based topology optimization of continuum structures. Stolpe and Svanberg introduced a linear mixed 0-1 model for topology optimization problems and sequential integer programming methods were used for the stress constrained topology optimization [17,18]. Li, Steven, et al. used ESO (evolutionary structural optimization) method for fully stressed and multi-criteria design [3,19,20].

In the present study, a level set based method is introduced for the topology optimization of continuum structures with stress constraints. Level set method, first introduced by Osher and Sethian in 1988 [21], is a method for numerical
simulation of motion of interfaces in two or three dimensions, and it has caught much attention in the topology optimization since the seminal papers [22-26]. Level set based method for the topology optimization is a boundary variation method where a structure is directly parametrized by its boundary and the free boundary of a structure is modified during the optimization, thus it maintains the 0-1 nature of topology optimization [27].

The level set method has been applied for stress-based optimization problems. A level set based method is proposed by Allaire and Jouve to minimize a domain integral of stress subject to material volume constraint [4]. Using X-FEM and level set description, Miegroet and Duysinx studied the shape optimization for minimizing stress concentration of 2D filets [28]. Our present study, being different to the above two studies, considers the stress constrained topology optimization of continuum structures.

2. Topology optimization problem with pointwise stress constraints

2.1 Problem definition

A shape $\Omega$ is an open bounded set occupied by isotropic linear elastic solid material. The Lipschitz continuous boundary of $\Omega$ consists of three disjoint parts

$$\partial \Omega = \Gamma_N \cup \Gamma_D \cup \Gamma_H$$

where a Dirichlet boundary condition is imposed on $\Gamma_D$, a Neumann boundary condition on $\Gamma_N$ and homogeneous Neumann boundary condition, i.e., traction free, on $\Gamma_H$. $\Gamma_H$ is the only part subject to optimization, i.e., the only part free to move during optimization. It is assumed that admissible shapes $\Omega$ stays in a fixed reference domain $D$.

The displacement field $u$ is the unique solution of the linear elasticity system

$$a(u, v) = \ell(v), \forall v \in U$$

Where

$$a(u, v) = \int_{\Omega} A e(u) : e(v) \, dx, \quad \ell(v) = \int_{\Omega} f v \, dx + \int_{\Gamma_H} t v \, ds$$

$f$ is the body force; $t$ is the boundary traction force; $\sigma = A e(u)$ is the stress tensor, $e(u)$ is the strain tensor, $U$ is the space of kinematically admissible displacement fields.

The optimization problem of the present work is to minimize the volume of the shape subject to stress constraints at each point, i.e.,

$$g(\sigma_v(x)) = \frac{\sigma_v(x)}{\sigma_a} \leq 0, \forall x \in \Omega$$

Where $\sigma_a$ is a pre-specified admissible stress; $\sigma_v$ is the von Mises stress, and in 2D it is

$$\sigma_v = \sqrt{\sigma_{xx}^2 + \sigma_{yy}^2 - \sigma_{xx}\sigma_{yy} + 3\sigma_{xy}^2}$$

According to eq. (2), a stress constraint is imposed at each point in the shape, although there is no conceptual difficulty in imposing the constraint only at several isolated points or in a subset of the shape.

Finally, we define the optimization problem as

$$\min J(\Omega) = \int_{\Omega} dx$$

s.t. $a(u, v) = \ell(u), \forall v \in U$

$$g(\sigma_v(x)) \leq 0, \forall x \in \Omega$$

2.2 Treatment of the pointwise stress constraints

The optimization problem given by eq.(4) are numerically challenging as there are a large number of stress constraints. An idea for overcoming this numerical challenge is to aggregate the stress constraint at each point into a single global constraint. For such purpose, Yang and Chen [2] investigated the Kreisselmeier-Steinhauser (KS) function [29] and the $p$-norm of local stress constraints; Duysinx and Sigmund utilized the $p$-norm and $p$-mean of local stress constraints [5]. Paris et al. compared the approaches of local and aggregated approaches [8].

There are two issues of aggregation of constraints that need to be considered. First, the equivalence between the pointwise constraints and the single global constraint. Second, the stability of convergence of the optimization. For the KS function, $p$-norm, and $p$-mean, it is well-known that the pointwise constraints and the global constraint are equivalent only when the so-called aggregation parameter is positive infinity [2,5,8]. However, this would leads to convergence difficulty. Particularly, when the aggregation parameter gets too large, the optimization problem becomes ill-conditioned [9], and the convergence of the optimization becomes oscillatory or even diverges [2,5,8]. Therefore, the aggregation parameter is usually set to a reasonably big value for the purpose of good stability of convergence. In such situation, the pointwise constraints and the single global constraint are not necessarily entirely equivalent.

In our present study, we propose a global constraint that aggregates the pointwise stress constraints as
\[ P(\sigma_v) = \frac{1}{2} \int_\Omega [\max(g(\sigma_v),0)]^2 \, dx \leq 0 \]

where \( \max(g,0) \) is important since it locates active stress constraints and prevents inactive stress constraints from coming into the global constraint [6,7]. One can see that when this global constraint is satisfied the pointwise constraints given by eq.(2) are guaranteed to be satisfied, and vice versa. Now, with the global stress constraint, the original optimization problem is converted to

\[
\begin{align*}
\min & \quad J(\Omega) = \int_\Omega dx \\
\text{s.t.} & \quad a(u,v) = \ell(u), \forall v \in U \\
& \quad p(\sigma_v) \leq 0
\end{align*}
\]

(5)

3. Sensitivity analysis

In order to solve the optimization problem eq.(9) via a gradient based method, with the free boundary of a shape constituting the design variable, it is required to find the effect of a shape variation on the variations in functions. In order to find such effect, it is convenient to use the material derivative and adjoint method.

Lagrangian of the optimization problem is given by

\[ L = J + a(u, w) - \ell(w) + \lambda p(\sigma_v) \]

where \( w \) is a Lagrange multiplier for the state equation of linear elasticity eq.(1); \( \lambda \) is a Lagrange multiplier for the global stress constraint. The material derivative of the Lagrangian is given by

\[ \dot{L} = \dot{J} + \dot{a}(u, w) - \dot{\ell}(w) + \lambda \dot{p}(\sigma_v) \]

(6)

where

\[
\begin{align*}
\dot{J} &= \int_\Omega \dot{V}_n \, ds \\
\dot{a}(u, w) &= \int_\Omega A \dot{e}(u') \cdot \dot{e}(w) \, dx + \int_\Omega A \dot{e}(u) \cdot \dot{e}(w') \, dx + \int_\Omega A \dot{e}(u) \cdot \dot{e}(w) V_n \, ds \\
\dot{\ell}(w) &= \int_\Omega f w' \, dx + \int_\Gamma f w V_n \, ds + \int_\Omega t w' V_n \, ds + \int_\Omega (\nabla (tw)^T \cdot n) \, ds
\end{align*}
\]

(7)

And \( V_n \) is the projection of velocity \( V \) onto the outward normal \( n \).

The material derivative of the aggregated constraint is given by

\[ \dot{p}(\sigma_v) = \int_\Omega \max(g,0) g'(\sigma_v) \, dx + \frac{1}{2} \int_\Omega [\max(g,0)]^2 V_n \, ds \]

(8)

Where the partial derivative of \( g \) is given by

\[ g'(\sigma_v) = \frac{\partial g}{\partial \sigma_v} \frac{\partial \sigma_v}{\partial \sigma_y} \frac{\partial \sigma_y}{\partial \sigma_x} (u') = \frac{1}{\sigma_v^2} [2 \sigma_{xx} - \sigma_{yy}, 2 \sigma_{yy} - \sigma_{xx}, 6 \sigma_{xy}] \sigma(u') \]

Substitute eq.(7)-(8) into eq.(6), we obtain the material derivative of the Lagrangian. Collecting all the terms that contain \( w' \) in the material derivative of the Lagrangian and letting the sum of these terms to be zero, we recover the weak form of the state equation (1). Collecting all the terms that contain \( u' \), and letting the sum of these terms to be zero, we obtain the adjoint equation, that is

\[ \int_\Omega A \dot{e}(u') \cdot \dot{e}(w) \, dx = -\dot{\lambda} \int_\Omega \max(g,0) g'(\sigma_v) \, dx, \forall u' \in U \]

(9)

Finally, collecting all the terms that contain \( V_n \) assuming there exists no body force, i.e., \( f=0 \), and noticing that only \( \Gamma_H \) subjects to optimization, we obtain the shape derivative of the Lagrangian as

\[ L' = \int_{\Gamma_H} G V_n \, ds, \quad G = 1 + A \dot{e}(u) \cdot \dot{e}(w) + \frac{1}{2} \dot{\lambda} [\max(g,0)]^2 \]

4. The level set method

Let \( \Omega \subseteq \mathbb{R}^d \) be the region occupied by a structure. The level set model specifies the boundary \( \Gamma \) in an implicit form as the zero level set of a one-higher dimensional scalar function, i.e.,

\[ \Gamma = \{ x \mid \Phi(x) = 0 \} \]

Then the design domain \( D \) is partitioned according to the following rule:

\[ \Phi(x) = 0 \Leftrightarrow \forall x \in \partial \Omega \cap D \]
\[ \Phi(x) < 0 \Leftrightarrow \forall x \in \Omega \setminus \partial \Omega \]
\[ \Phi(x) > 0 \Leftrightarrow \forall x \in D \setminus \Omega \]
Where $D$ is a fixed design domain in which all admissible shapes $\Omega$ are included, i.e. $\Omega \subseteq D$. In the level set method, the scalar function $\Phi$ is generally constructed to be a signed distance function ($\Phi = 1$) to the boundary. With such a signed distance function, the unit outward normal $n$ to the boundary is $n = \nabla \Phi$.

Propagation of the free boundary of a structure in the optimization process is described by the Hamilton-Jacobi equation:

$$
\Phi_t + V_n = 0
$$

(10)

where $V_n$ is the velocity vector defined on the boundary. Usually, a proper variation of the free boundary of a structure is obtained as a descent direction of an objective function via sensitivity analysis. Then, such boundary variation is treated as the velocity $V$ in the Hamilton-Jacobi equation.

It should be noted that in the level set based method the design velocity $V_n$ defined on the free boundary $\Gamma_H$ of a structure must be extended to the whole reference domain $D$ or a narrow band around the free boundary [31]. In the topology optimization of continuum structures, since a fixed mesh is used for the finite element analysis, and voids are represented by artificial weak material, the normal velocity which is defined on the traction free boundary $\Gamma_H$ can be naturally extended to the entire reference domain $D$ as $V_n = V_n$ for all $x$ in $D$. This extended velocity will, however, introduce discontinuity of the velocity at the neighborhood of the free boundary since the strain field is not continuous across the free boundary. To guarantee a smooth propagation of the free boundary, this discontinuity should be eliminated. Hence, a smoothing operation is introduced for the velocity extension [30].

The HJ equation eq.(10) is a hyperbolic type of PDE. A variety of spatial and time discretization schemes were devised to solve this type of PDE. In the present study, the first order upwind spatial differencing and forward Euler time differencing are utilized. Finally, since the level set function $\Phi$ often becomes too flat or too steep during optimization which leads to increasing numerical error, a reinitialization procedure is periodically performed to restore $\Phi$ to a signed distance function to the structural boundary. More details can be found in [31].

5. Numerical implementation

In the level set based method, when the boundary of a structure is moving across an element, remeshing, which can be highly labor intensive, cannot be avoided if accurate analysis using the standard finite element method is required. To enhance the efficiency of FEA, Eulerian-type method, employing a fixed mesh and artificial weak material, is adopted as an alternative finite element analysis tool in the present study. In our implementation, because the grid for computation of level set coincides with the fixed finite element mesh, we treat an element as solid if the level set function at the center of the element is smaller than zero, and as void if bigger than zero. Thus, intermediate materials that would lead to incorrect stress constraint will not arise around the boundary. A further fact needing to be pointed out is that with such definition of solid and void, the resulting structure will have a staircase-shaped boundary that is similar to that obtained by the SIMP method. One can refine the grid to achieve high smoothness of structure boundary, or one can reconstruct a smooth boundary in a post-process stage.

For computing the adjoint force of the adjoint equation, eq.(9) Gauss integration is used. When a Gauss point does not lie in the shape, there exists no adjoint force at the Gauss point. In other word, the stress constraint is not considered for the void in the reference domain.

It is well known that in the vicinity around the point where a concentrated load is applied a stress singularity occurs. In the present work, such singular stress is not considered as constraint violation and we impose no pointwise stress constraints in a number of elements near such stress singular point.

6. Numerical examples

In this section the proposed level set based method is applied to several examples in two dimensions, although there exists no conceptual difficulty to do the numerical examples in three dimensions. In these examples, it is assumed that the solid material has a Young's modulus $E=1$, and Poisson's ratio $\nu=0.3$; the artificial weak material has a Young's modulus $E=0.001$ and Poisson's ratio $\nu=0.3$. For all the examples, a fixed mesh of 4-node bilinear square elements is used, the adjoint force in the adjoint equation is obtained via 4-point Gaussian integration, and the plane stress statement is assumed. All the CPU time is based on a computer with an Intel Core2 Duo P8600 processor of 2.4 GHz clock speed, 2G memory, and Matlab 7.7.0.

6.1 A L-shape beam

The optimal design problem of a L-shape beam is shown in fig.1. The thickness of the L-shape beam is 1 unit vertical point load $t=1$ applied at the middle point of the right side. 6400 4-node bilinear square elements are used for the finite element analysis. A $101 \times 101$ rectilinear grid of the square which contains the L-shape polygon is used for level set computations. A fixed Lagrange multiplier $\lambda=20$ is applied for the first 300 iterations, then it is updated according to the augmented Lagrange multiplier method. The admissible von Mises stress is $\sigma_a=40$. The initial design is shown in fig. 1(b), and the red rectangle shows the neighborhood around the point load where stress constraints are not considered. After each finite element analysis, we perform 2 time steps of the Hamilton-Jacobi equation, and a reinitialization is performed.
In the initial design of the L-shaped beam, stress concentration occurs at the inner corner which requires fine or adaptive finite element mesh to capture it. However, in order to simplify the finite element analysis and reduce the computational effort, we use a fixed mesh and artificial weak material as an alternative finite analysis tool. From the results of numerical experiments using different finite element meshes, we see that the mesh we used is sufficient for this example and the optimization is not numerically affected by the stress singularity.

In the optimal stress design shown in fig. 2(a), one can see that the boundary at the inner corner is smoothed and rounded, and there is no stress concentration near the inner corner as can be seen in fig. 2(b). Also, one can see in fig. 2(b) that only at several boundary point around the inner corner the von Mises stresses are larger than the admissible stress. In fact, these large stresses occur merely due to the rough boundary as can be seen in fig. 2(a) but not due to the improper shape and topology of the structure, thus it is not an indication of failure of the stress constrained optimization.

The convergence history of the optimization is shown in fig. 3. At the convergence, the value of the global stress constraint is $3.43 \times 10^{-4}$; the compliance is 262.41; the volume of material is 0.54. The computation time is 40 seconds for each optimization step on average and 7 hours at the convergence. As one can see in fig. 3, the value of the global stress constraint oscillates during the first 200 iterations. The reason for this oscillation is explained as follows. During the first 200 iterations, boundary of the structure undergoes drastic topological changes (split apart or merge together), and large stresses occur in the neighborhood of such boundary because there exists sharp corners. This fact has two implications for the implementation of the present level set based method. First, for the first several hundred iterations, the Lagrange multiplier should not set too big, otherwise the meaningless large stress due to occurrence of topological changes will unduly dominates the optimization and driven it to a wrong way. Second, the step size of each iteration should not be too big; otherwise the oscillation will be even bigger.

The volume of material of the optimal structure is used for a new optimization problem where the compliance is mi-
The setting of FEM and level set is kept unchanged, and the initial design is the same as the one shown in fig.1(a). The computation time is 4 seconds for each optimization step on average and 0.3 hour at the convergence. The results are shown in fig.4. At the convergence, the value of the global stress constraint is $1.49 \times 10^{-4}$; the compliance is 157.60; the volume of material is 0.54.

![Figure 4: Results of compliance design. (a) optimal structure, (b) von Mises stress.](image)

From the results of the L-shape beam example, we can see that the stress design is quite different from the compliance design, especially near the inner corner. One can see the stress concentration at the inner corner of the compliance design as shown in fig. 4(a). Such results indicate that the compliance criterion is indeed incapable to take care of the local stress near the inner corner.

During the optimization process, no singularity phenomena were found. Moreover, from the optimal stress designs, contour maps of von Mises stress, and the convergence histories, the optimization indeed converges, but on special optimization algorithm, such as $\varepsilon$-relaxation method was used. Thus, we think that the present level set based method is convenient for the stress-constrained topology optimization of continuum structures.

### 6.2 A MBB beam

The optimal design problem of a MBB beam is shown in fig.5. The whole design domain D is a rectangle of size $6 \times 1$ and thickness 1, with a unit vertical point load $t=1$ being applied at the middle point of the top side. A $120 \times 40$ mesh is used for the finite element analysis of a half of the MBB beam; a $121 \times 41$ rectilinear grid is used for level set computations. A fixed Lagrange multiplier $\lambda=25$ is applied for the first 300 iterations, then it is update according to the augmented Lagrange multiplier method. The admissible von Mises stress is $\sigma_a=15$. The initial design is shown in fig.5(b). We perform two time steps of the Hamilton-Jacobi equation after each finite element analysis, and a reinitialization is performed after each optimization step. In this numerical example, no pointwise stress constraints are imposed in the neighborhood of the point load and the supports where singular stress occurs shown in fig.5(b) as the red rectangles.

![Figure 5: Design problem and initial design of a MBB beam. (a) optimal structure, (b) von Mises stress.](image)

The results are shown in fig.6, and the convergence history is shown in fig.7. At the convergence, the value of the global stress constraint is $4.53 \times 10^{-4}$; the compliance is 186.00; the volume of material is 3.54. The computation time is 12 seconds for each optimization step on average and 1.7 hour at the convergence. As we can see from fig.6(a), due to bending at the center region of the MBB beam, much material is accumulated during the optimization at the region to reduce the stress.

The volume of material of the structure is used for a new optimization problem where the compliance is minimized. The setting of FEM and level set is kept unchanged, and the initial design is the same. The computation time is 6 seconds for each optimization step on average and 0.3 hour at the convergence. The results are shown in fig.8. At the convergence, the value of the global stress constraint is $6.40 \times 10^{-4}$; the compliance is 165.16; the volume of material is 3.54.

Comparing fig.6(a) and fig.8(a), one can see notable difference of shape and topology of the present stress design and...
compliance design, although both of the two designs are just locally optimal. Also, one can see that the von Mises stress at the center region of the compliance design is bigger than $\sigma_a = 15$. Also, the value of the aggregated stress constraint of the compliance design is bigger than that of the stress design.

Figure 6: Results of stress design with $\sigma_a = 15$.

![Figure 6](image1)

Figure 7: Convergence history of the stress design with $\sigma_a = 15$.

Figure 8: Results of compliance design. (a) optimal structure, (b) von Mises stress.

![Figure 8](image2)

7. Conclusions
In the present study, a level set based method is proposed for the topology optimization of continuum structures with stress constraint. Pointwise stress constraints are aggregated into a single equivalent global constraint. Having the global constraint, a gradient based optimization algorithm requires only the shape sensitivity of the global constraint and no longer requires those of individual stress constraints. The shape sensitivity was derived using the material derivative and adjoint method. Several numerical examples were provided in two dimensions.

Using the level set based method that essentially complies with the 0-1 nature of topology optimization, one does not need to deal with several issues concerning the stress constraint that would occur if the density based SIMP method is used, in particular the stress criteria of microscale and stress singularity, hence giving a convenient alternative for the stress constrained optimization.

Acknowledgments
This research work is partly supported by the Special Fund for Basic Research Work in Central Universities (Grant No. Q2009009), Natural Science Foundation of Hubei Province (Grant No. 2009CDB321).

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