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A simple proof of the almost sure discreteness of a class of random measures

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Abstract

A simple proof of the almost sure discreteness of a class of random measures which includes completely random measures is presented. The method of proof shows how one may extend the specific argument of Berk and Savage (Berk and Savage Contributions to Statistics, Jaroslev Hajek Memorial Volume, Reidel, Dordrecht, Boston, Massachusetts, London, 25 (1979)) and Lo and Weng (Ann. Inst. Statist. Math. 41 (1989) 227), for the Dirichlet and weighted gamma processes, respectively. The technique is based on a disintegration argument which reveals the role of a necessary positivity condition.

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Let N denote a Poisson process on an arbitrary Polish space $\mathscr{W}=\mathscr{S}\times\mathscr{Y}$ with mean intensity

$$E[N(\mathrm{d}s,\mathrm{d}y)]=v(\mathrm{d}s,\mathrm{d}y)=\rho(\mathrm{d}s|y)\eta(\mathrm{d}y)$$

for ρ a Lévy measure on the Polish space $\mathscr{S}$ depending on y in a fairly arbitrary way and η a σ-finite (non-atomic) measure on $\mathscr{Y}$. Let $\mathrm{BM}_+(\mathscr{W})$ denote the class of bounded measurable non-negative functions on $\mathscr{W}$. The distribution of N is characterized by its Laplace functional defined for functions $g\in\mathrm{BM}_+(\mathscr{W})$ as

$$\mathscr{L}_N(g|\rho,\eta)=\exp\left(-\int_{\mathscr{Y}}\int_{\mathscr{S}}(1-\mathrm{e}^{-g(s,y)})\rho(\mathrm{d}s|y)\eta(\mathrm{d}y)\right). \tag{1}$$

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Denote the law of N by $\mathscr{P}(\mathrm{d}N|\rho,\eta)$. Let h denote an arbitrary strictly positive function on $\mathscr{S}$. Furthermore, it is assumed that h,ρ,η are selected such that for each bounded set B in $\mathscr{Y}$,

$$\int_B \int_{\mathscr{S}} \min(h(s),1)\rho(\mathrm{d}s|y)\eta(\mathrm{d}y) < \infty. \tag{2}$$

Now define a random measure μ on $\mathscr{Y}$ such that it may be represented in a distributional sense as

$$\mu(\mathrm{d}y) = \int_{\mathscr{S}} h(s)N(\mathrm{d}s,\mathrm{d}y). \tag{3}$$

Representation (3), coupled with (1), implies that the Laplace functional for μ can be represented as

$$\mathscr{L}_\mu(f|\rho,\eta) = \exp\left(-\int_{\mathscr{Y}}\int_{\mathscr{S}}(1-\mathrm{e}^{-f(y)h(s)})\rho(\mathrm{d}s|y)\eta(\mathrm{d}y)\right)$$

for f in a space of non-negative bounded measurable functions, $\mathrm{BM}_+(\mathscr{Y})$. That is to say the Laplace functional is obtained by setting, $g(y,s)=f(y)h(s)$ in (1). Denote the law of μ by $\mathscr{P}(\mathrm{d}\mu|\rho,\eta)$. Following Daley and Vere-Jones (1988), condition (2) guarantees that μ is in the space of boundedly finite measures $\mathscr{M}$ equipped with an appropriate sigma-field, $\mathscr{B}(\mathscr{M})$. In the case that $\mathscr{S}=\mathfrak{R}^+$ and $h(s)=s$, then following Kingman (1967, 1993), μ is called a completely random measure without a deterministic component. The class of μ was first discussed in Perman, et al. (1992). James (2002) analyzes such processes within the context of problems related to Bayesian non-parametrics.

As is well known, there is a subtle difference between the equality in distribution of random measures and the behavior of their sample paths. This paper shows how a disintegration argument may be used to establish the almost sure discreteness of μ. In particular, the result provides a simple alternative proof for the class of completely random measures as discussed in Kingman (1993, Chapter 8). Kingman's result is based on a modification of Blackwell's (1973) argument for the Dirichlet process. The present technique is based on the approach of Berk and Savage (1979) and Lo and Weng (1989, Corollary 3.1) for the Dirichlet process and weighted gamma process, respectively. A streamlined version of the proof of Berk and Savage (1979) is discussed in Basu and Tiwari (1982). The Dirichlet process is not a completely random measure but may be defined as a normalized gamma process. The main interest is to show how one may extend their arguments which relied on the fact that the conditional distributions of a Dirichlet process evaluated at a point y, say $P(\{y\})|y$, and a gamma process, say $G(\{y\})|y$, are continuous beta and gamma random variables, respectively for almost all points y. Such a result for more general μ, including other completely random measures, is not immediately obvious. Consider the joint product measure of (y,μ),

$$\mu(\mathrm{d}y)\mathscr{P}(\mathrm{d}\mu|\rho,\eta). \tag{4}$$

Note this is not a proper joint distribution. Here it is shown that what is required is either μ has a sigma-finite first moment measure,

$$m_\mu(\mathrm{d}x|\rho,\eta) = \int_{\mathscr{M}} \mu(\mathrm{d}x)\mathscr{P}(\mathrm{d}\mu|\rho,\eta) = \left[\int_{\mathscr{S}} h(s)\rho(\mathrm{d}s|x)\right]\eta(\mathrm{d}x), \tag{5}$$

where $\int_{\mathscr{S}} h(s)\rho(\mathrm{d}s|x) < \infty$ for almost all x, or the law of μ is absolutely continuous to the law of another measure μ^* which has a first moment measure. Following the proof of the main result, it will be shown how the absolute continuity criterion can be applied for the case of the stable law process. As shown in James (2002), condition (5) allows one to describe a disintegration of the joint measure (4) which, as shall be shown, facilitates a simple proof of the discreteness of μ analogous to the cases for the Dirichlet and gamma processes. For clarity, we will first discuss how the disintegration of (5) can be easily obtained via a Laplace functional exponential change of measure formula, discussed in James (2002). The formula, as shown in James (2002), has a variety of applications. James (2003a,b) discuss how it is used in applications to Lévy moving average models and a class of spatial neutral to the right processes. The formula is analogous to, but much more general than, Proposition 3.1 of Lo and Weng (1989) in the case of the weighted gamma process. The result is now described below in Proposition 1. The proof of Proposition 1 given in James (2002, 2003a,b) is also provided to demonstrate its simple derivation. Here, v is an arbitrary non-atomic measure.

Proposition 1 (James, 2002). *For* $g \in \mathrm{BM}_+(\mathscr{W})$ *and each* τ *on* $(\mathscr{M}, \mathscr{B}(\mathscr{M}))$

$$\int_{\mathscr{M}} \tau(N)\mathrm{e}^{-N(g)}\mathscr{P}(\mathrm{d}N|v) = \mathscr{L}_N(g|v)\int_{\mathscr{M}} \tau(N)\,\mathscr{P}(\mathrm{d}N|\mathrm{e}^{-g}v),$$

where $\mathscr{P}(\mathrm{d}N|\mathrm{e}^{-g}v)$ *is the law of a Poisson process with intensity* $\mathrm{e}^{-g(w)}v(\mathrm{d}w)$. *In other words, the following absolute continuity result holds*, $\mathrm{e}^{-N(g)}\mathscr{P}(\mathrm{d}N|v) = \mathscr{L}_N(g|v)\mathscr{P}(\mathrm{d}N|\mathrm{e}^{-g}v)$.

Proof. By the unicity of Laplace functionals for random measures on $\mathscr{W}$ it suffices to check this result for the case $\tau(N) = \mathrm{e}^{-N(l)}$, for $l \in \mathrm{BM}_+(\mathscr{W})$. It follows that

$$\int_{\mathscr{M}} \mathrm{e}^{-N(g+l)}\mathscr{P}(\mathrm{d}N|v) = \mathscr{L}_N(g|v)\int_{\mathscr{M}} \mathrm{e}^{-N(l)}\mathscr{P}_g(\mathrm{d}N),$$

where for the time being $\mathscr{P}_g$ denotes some law on N. Simple algebra shows that

$$\int_{\mathscr{M}} \mathrm{e}^{-N(l)}\mathscr{P}_g(\mathrm{d}N) = \frac{\mathscr{L}_N(g+l|v)}{\mathscr{L}_N(g|v)},$$

and hence $\mathscr{P}_g(\mathrm{d}N) = \mathscr{P}(\mathrm{d}N|\mathrm{e}^{-g}v)$. □

Note that setting $g(s,y) = f(y)h(s)$ in Proposition 1 yields immediately the following absolute continuity formula for μ, which equates to a functional exponential tilting operation for different f. See also Proposition 3.1 in James (2002).

Corollary 1. *Let* μ *be* $\mathscr{P}(\mathrm{d}\mu|\rho,\eta)$, *then for* $f \in \mathrm{BM}_+(\mathscr{Y})$ *the following absolute continuity relationship holds*:

$$\mathrm{e}^{-\mu(f)}\mathscr{P}(\mathrm{d}\mu|\rho,\eta) = \mathscr{P}(\mathrm{d}\mu|\mathrm{e}^{-fh}\rho,\eta)\mathscr{L}_\mu(f|\rho,\eta).$$

Now, we show, that under (5), how Proposition 1 or Corollary 1 implies the following disintegration formulae for μ,

Proposition 2. *Let* μ *be* $\mathscr{P}(\mathrm{d}\mu|\rho,\eta)$ *such that* μ *has a* 1*st moment measure* $m_\mu(\cdot|\rho,\eta)$, *then* (4) *admits the disintegration*

$$\mu(\mathrm{d}x)\mathscr{P}(\mathrm{d}\mu|\rho,\eta)=\mathscr{P}(\mathrm{d}\mu|\rho,\eta,x)m_\mu(\mathrm{d}x|\rho,\eta).$$

$\mathscr{P}(\mathrm{d}\mu|\rho,\eta,x)$ *denotes a conditional distribution of* μ *given a point* x *which is equivalent in law to the random measure*

$$\mu(\cdot)+h(J)\delta_x(\cdot), \tag{6}$$

where μ *is* $\mathscr{P}(\mathrm{d}\mu|\rho,\eta)$ *independent of* $h(J)$, *and* J *is a random element in* $\mathscr{S}$ *with distribution* $\propto h(s)\rho(\mathrm{d}s|x)$ *for almost all* x.

Proof. The result is equivalent to the equality

$$\int_{\mathscr{Y}}\int_{\mathscr{M}} g(x,\mu)\mu(\mathrm{d}x)\mathscr{P}(\mathrm{d}\mu|\rho,\eta)=\int_{\mathscr{Y}}\int_{\mathscr{M}} g(x,\mu)\mathscr{P}(\mathrm{d}\mu|\rho,\eta,x)m_\mu(\mathrm{d}x|\rho,\eta)$$

for g arbitrary measureable positive or integrable functions on $\mathscr{Y}\times\mathscr{M}$. It suffices to check this for $g(x,\mu)=t(x)\mathrm{e}^{-\mu(f)}$, for $t,\ f\in \mathrm{BM}_+(\mathscr{Y})$. Noting that

$$\int_{\mathscr{M}}\mu(\mathrm{d}x)\mathscr{P}(\mathrm{d}\mu|\mathrm{e}^{-fh}\rho,\eta)=\left[\int_{\mathscr{S}} h(s)\mathrm{e}^{-f(x)h(s)}\rho(\mathrm{d}s|x)\right]\eta(\mathrm{d}x),$$

an application of Proposition 1 or Corollary 1 yields

$$\int_{\mathscr{M}}\left[\int_{\mathscr{Y}} t(x)\mu(\mathrm{d}x)\right]\mathrm{e}^{-\mu(f)}\mathscr{P}(\mathrm{d}\mu|\rho,\eta)=\mathscr{L}_\mu(f|\rho,\eta)\int_{\mathscr{Y}} t(x)\left[\int_{\mathscr{S}}\mathrm{e}^{-f(x)h(s)}h(s)\rho(\mathrm{d}s|x)\right]\eta(\mathrm{d}x).$$

Multiplying and dividing by $\int_{\mathscr{S}} h(s)\rho(\mathrm{d}s|x)$, identifies the appropriate conditional Laplace functional of μ given x as

$$\mathscr{L}_\mu(f|\rho,\eta)\int_{\mathscr{S}}\mathrm{e}^{-f(x)h(s)}\frac{h(s)\rho(\mathrm{d}s|x)}{\int_{\mathscr{S}} h(s)\rho(\mathrm{d}s|x)}. \qquad\square$$

Remark 1. Disintegrations of this type are well defined on Polish spaces and fit within the theory of Palm calculus as described for instance in Daley and Vere-Jones (1988, Chapter 12). James (2002, 2003a) establish more general versions of Proposition 2 based on representations in terms of partitions of the integers $\{1,\dots,n\}$.

The discreteness result will now be presented. For the arguments below it suffices to show that the result holds over all bounded sets B, i.e. sets such that $\eta(B)<\infty$ so without loss of generality it will be assumed that η is a finite measure.

Theorem 1. *Suppose that* μ *is* $\mathscr{P}(\mathrm{d}\mu|\rho,\eta)$ *such that* μ *has a* 1*st moment measure* $m_\mu(\cdot|\rho,\eta)$. *Otherwise suppose that there exists a measure* μ^*, *which admits a* 1*st moment measure and satisfies the absolute continuity relationship*

$$g(\mu)\mathscr{P}(\mathrm{d}\mu|\rho,\eta)=\mathscr{P}(\mathrm{d}\mu^*|\rho^*,\eta), \tag{7}$$

for some positive integrable function g. Then μ is almost surely discrete. That is,

$$\int_{\mathscr{M}} \mu(\{x : \mu(\{x\}) = 0\})\mathscr{P}(\mathrm{d}\mu|\rho,\eta) = 0.$$

Proof. Using Proposition 2

$$\int_{\mathscr{M}} \mu(\{x : \mu\{x\} = 0\})\mathscr{P}(\mathrm{d}\mu|\rho,\eta) = \int_{\mathscr{Y}} \left[\int_{\mathscr{M}} I\{x : \mu(\{x\}) = 0\}\mathscr{P}(\mathrm{d}\mu|\rho,\eta,x)\right] m_\mu(\mathrm{d}x|\rho,\eta).$$

From (6), the law of μ taken with respect to the inner term (that is given x) is the same as

$$\mu(\cdot) + h(J)\delta_x(\cdot),$$

where $h(J)$ is strictly positive and J has law $\propto h(s)\rho(\mathrm{d}s|x)$ for almost all x. But since μ is not negative it follows that

$$\mu(\{x\}) + h(J)\delta_x(\{x\}) \geqslant h(J) > 0,$$

for almost all x with respect to m_μ. Hence the inner term is zero which concludes the result. If again μ does not admit such a disintegration, apply the result to the random measure μ^*, and use the absolute continuity of measures. □

The proof of the result shows that the strict positivity of the function h guarantees the almost sure discreteness. Recall that in the case of completely random measures, $h(s) = s \in (0,\infty)$. The absolute continuity condition (7) can be established quite easily. One may use Corollary 1 for a convenient form of f. This is demonstrated for the stable law with index $0 < \alpha < 1$. In that case, the Lévy measure is of the form

$$\rho_\alpha(\mathrm{d}s) = \frac{1}{\Gamma(1-\alpha)} s^{-\alpha-1}\,\mathrm{d}s, \quad \text{for } s > 0$$

and hence $\int_0^\infty s\rho_\alpha(\mathrm{d}s) = \infty$. However, using Corollary 1 for $f \in \mathrm{BM}_+(\mathscr{Y})$, it follows that the stable law random measure is absolutely continuous to a weighted generalized gamma random measure whose Lévy measure is defined as

$$\rho_{\alpha,f}(\mathrm{d}s|x) = \frac{1}{\Gamma(1-\alpha)} s^{-\alpha-1} \mathrm{e}^{-f(x)s}\,\mathrm{d}s,$$

where $\int_0^\infty s\rho_{\alpha,f}(\mathrm{d}s|x) = f(x)^{\alpha-1}$.

Theorem 1 implies the almost sure discreteness of random measures defined as $g(\mu)\mu(\cdot)$ for a non-negative function g such that $g(\mu)\mu(\cdot)$ is integrable. This is demonstrated for the class of random probability measures on $\mathscr{Y}$ representable in distribution as

$$P(\cdot) = \frac{\mu(\cdot)}{\mu(\mathscr{Y})} \tag{8}$$

for $0 < \mu(\mathscr{Y}) < \infty$ almost surely. Denote the law of P as $\mathscr{P}(\mathrm{d}P|\rho,\eta)$. The Dirichlet process is defined when μ is specified as a gamma process with finite shape measure η. The important random probability measure based on a stable law process is also a member of this class. The result is provided in the next corollary.

Corollary 2. *The class of random probability measures defined in* (8) *are discrete almost surely. That is,*

$$\int_{\mathcal{M}} P(\{x : P(\{x\}) = 0\})\mathscr{P}(\mathrm{d}P|\rho,\eta) = 0.$$

Proof. The proof is immediate from Theorem 1 since,

$$\int_{\mathcal{M}} P(\{x : P(\{x\}) = 0\})\mathscr{P}(\mathrm{d}P|\rho,\eta) = \int_{\mathcal{M}} \mu(\{x : \mu(\{x\}) = 0\})\ \frac{1}{\mu(\mathscr{Y})}\mathscr{P}(\mathrm{d}\mu|\rho,\eta) = 0. \qquad \square$$

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