Determination of Effective Elastic Properties of Microcracked Rocks Based on Asymptotic Approximation

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Abstract

In this paper, we present a study on the effective elastic properties of finely fractured rock based on energy equivalence. For a rock body weakened by many penny-shaped microcracks, its effective elastic properties such as apparent Young's modulus, shear modulus and Poisson's ratio are particularly important to many applications. To determine these macroscopic parameters, we first adopted a dilute solution approach outlined in Kachanov (1992, App. Mech. Res.) where a crack compliance tensor is defined and used extensively to express the energy perturbation caused by cracks. In parallel, a self-consistent method is employed where an asymptotic form of the Eshelby's tensor (Eshelby, 1957, Proc. R. Soc. Lond. Ser. A) is developed to treat a penny-shaped microcrack as an extreme case of a spheroid-shaped inhomogeneity. The asymptotic approximation permits the local Eshelby tensor as well as its global expression to be derived analytically, and to be further used to construct an equivalent eigenstrain problem based on energy equivalence. The effective values for parameters of interest can then be evaluated. The formulation and results obtained through the asymptotic self-consistent approach are compared to those in the non-interacting scheme.

1 Introduction

One of the classic problems in solid mechanics, geophysics and material science is the determination of the effective elastic properties of cracked solids. It is particularly important for constitutive modelling of brittle microcracking materials such as rocks with large quantities of microcracks. The elastic properties of cracked rocks depend on a number of facts: the mineral properties and distribution, the porosity type, magnitude and distribution; and the state of saturation. There are two major theoretical approaches in the literature addressing the problem of effective elastic moduli of cracked rocks. The first class is the effective medium theories that assume the separate pores and cracks that may or may not connect in the rocks; the second theory is the poroelastic theory which assumes the significant portions of the pores and cracks are connected. The poroelastic theories were pioneered by the constitutive equations developed by Biot (1941) which are essentially phenomenological in nature and thus do not require characterisation of matrix and pore space geometry. In contrast, the effective medium theories generally require parameters characterising the pore shape and distribution, and can be traced back as early as to the classical bounds provided by Voigt (1928) and Reuss (1929) and Hashin and Shtrikman (1961, 1962). Pioneering works were motivated by microcracking in either metals or in rocks, and the effective constants were derived in the isotropic case of randomly oriented cracks of circular/penny shapes in the non-interaction approximation (Bristow, 1960; Walsh 1965a, b). A major flaw of these early non-interaction studies is that, when the crack densities are so high that interactions among cracks become significant, the non-interaction predictions will inevitably lose their accuracy. To account for crack interactions, numerous improved approximate schemes have been proposed (see Kachanov, 2007 for a latest review). Amongst them are the well-known self-consistent scheme (Budiansky and O'Connell, 1976), the differential method (Vavakin and Salganik, 1975, Norris, 1985; Hashin, 1988; Zimmerman, 1991), the Mori-Tanaka method (Mori and Tanaka, 1973; Benveniste, 1987), and the non-interacting method (Kachanov, 1992).

A common feature in most of these schemes is that they consider the interactions by placing non-interacting cracks into an "effective environment". Many early studies focus on a general solid with a matrix containing inclusions, from which the problem of effective elastic moduli of cracked solids can be regarded as a limiting case. In this limit case, it is assumed the inclusions shrink to surfaces and meanwhile their elastic moduli tend to zero. As remarked in a review by Kachanov (1992), however, there are some difficulties pertaining to this approach, such as the dependence of transition order of the limit, the definition of crack density parameter which can not take into account the crack shape, and the problems associated with some existing approaches in doing the degeneration. Predictions by these various schemes also diverge significantly as crack density increases. The self-consistent scheme gives strong softening effect of interactions, while the differential scheme gives rise to substantially milder softening effect. The Mori-Tanaka scheme for materials with interacting inclusions predicts no interaction effect at all when applied to cracked solids. As far as cracks are concerned, the seminal Hashin-Shtrikman (1963) bounds fail to provide any guidance. The Hashin-Shtrikman (1963) bound on the effective modulus κ_* of a three-dimensional two-phase composite is:

$$c_1\kappa_1 + c_2\kappa_2 - \frac{c_1c_2\left(\kappa_1-\kappa_2\right)^2}{c_1\kappa_2 + c_2\kappa_1 + 4\mu_2/3} \le \kappa_* \le c_1\kappa_1 + c_2\kappa_2 - \frac{c_1c_2\left(\kappa_1-\kappa_2\right)^2}{c_1\kappa_2 + c_2\kappa_1 + 4\mu_1/3}$$

where κ_1, κ_2, μ_1 and μ_2 ($\mu_2 \le \mu_1$) are the phase bulk and shear moduli, and c_1 and c_2 are the phase volume fractions. Now consider the case of cracks where the quantities κ_2, μ_2 and c_2 all tend to zero, then the upper bound in the Hashin-Shtrikman expression degenerates into the following trivial one:

$$\kappa_* \le \kappa_1$$

with the lower bound totally indeterminable. Comprehensive comparisons and comments of the various approaches in application to cracked solids can be referred to Kachanov (1992).

In view of the above, direct approaches have been suggested to solve the problem of effective moduli of cracked solids by Kachanov (1992). In this paper we employ two different approaches, one is the non-interacting scheme outlined in Kachanov (1992) based on a crack compliance tensor, and another by a self-consistent approach based on an asymptotic approximation of the Eshelby's tensor. We demonstrate that the two method can lead to the same results as developed by Budiansky and O'Connell (1976). Predictions of effective moduli by using the two methods are compared, as well as with an approximate analytical solution by differential scheme.

2 Non-interacting scheme based on crack compliance tensor

2.1 Crack density parameter

Budiansky and O'Connell (1976) defined the following 2D scalar crack density ρ for a microcracked solid as:

$$\beta = N\left\langle l^2\right\rangle \tag{1}$$

where N is the number of microcracks per unit area in the plane, l is the half length of a 2D slit-like microcrack, and $\left\langle l^2\right\rangle$ stands for the average of l^2 over all microcracks. The matrix is assumed to be isotropic. Kachanov (1980, 1987) defined the following symmetric second-order crack density tensor for the general case of 3D solids with penny-shaped cracks:

$$\boldsymbol{\alpha} = \frac{1}{V}\sum_k \gamma_k \mathbf{n}_k \mathbf{n}_k \tag{2}$$

Where V is the volume of averaging, $\mathbf{n}_k$ is the unit normal to the kth crack and γ_k is a weighting factor characterizing the contribution of the kth crack to $\boldsymbol{\alpha}$ and depending on the physical problem of interest. In the case of effective properties $\gamma_k = r_k^3$, where r_k is the radius of the kth crack for the 3D solids with penny-shaped cracks. For a 2D solid with slit-like cradcks, $\gamma_k = l_k^2$ where l_k is the half-length of the kth crack and V should be changed to the area of averaging. Note that the trace of $\boldsymbol{\alpha}$, $\rho = tr\boldsymbol{\alpha}$, coincides with the scalar crack density as defined in Eq.(1) for a general 2D case, and with the scalar crack density $\rho = N\left\langle a^3\right\rangle$ for a general case, which implies $\boldsymbol{\alpha}$ represents a tensorial generalisation of ρ accounting for the crack orientation statistics. The crack density plays a role roughly similar to the role of the volume fraction for two-phase composites. Note that similar expression as Eq.(2) has been used as damage tensor for crack rocks, see, e.g., Zhao et al. (2005) and Zhao and Sheng (2006).

2.2 Crack compliance and the non-interacting scheme

Kachanov (1992, 1994) introduced a crack compliance tensor to characterise the crack open displacement (COD)of a crack in a matrix under external force. A similar expression was first used by Hill (1963) for a more general case of arbitrary cavities. It is assumed that, across the crack surfaces the displacements are discontinuous such that the strains are singular, which can be denoted by the following expression:

$$\varepsilon_{ij} = M^0_{ijkl}\sigma_{kl} + \frac{1}{2}\sum_k \left(b_i n_j + b_j n_i\right)^{(k)} \delta\left(S^{(k)}\right) \tag{3}$$

where the first term in the right-hand side, $M^0_{ijkl}\sigma_{kl}$, denotes the regular part with M^0_{ijkl} being the linear elastic compliance tensor of the matrix. $\delta(\cdot)$ is a delta function concentrated on the k th crack's surfaces $s^{(k)}$. $\mathbf{b} = \mathbf{u}^+ - \mathbf{u}^-$ is the vector for displacement jump across the crack surfaces (or crack opening displacement, COD). And $\mathbf{n}$ the unit normal to a crack. Both $\mathbf{n}$ and $\mathbf{b}$ are generally variable along cracks. Averaging the above expression over the volume of the material body V and using the property of $\delta(\cdot)$ that the integral $\int_V f(x)\delta(s)dV$ can be reduced to the surface integral $\int_S f(x)ds$, we have:

$$\begin{aligned}\left\langle \varepsilon_{ij}\right\rangle &= M^0_{ijkl}\left\langle \sigma_{kl}\right\rangle + \frac{1}{2V}\sum_k \int_{S^{(k)}} \left(b_i n_j + b_j n_i\right)^{(k)} ds \\ &= \left(M^0_{ijkl} + \Delta M_{ijkl}\right)\left\langle \sigma_{kl}\right\rangle = M_{ijkl}\left\langle \sigma_{kl}\right\rangle\end{aligned} \tag{4}$$

where ΔM_{ijkl} is the change in compliance due to cracks and M_{ijkl} is the effective compliance.

It is assumed that the material body contains a statistically homogeneous field of cracks and is sufficiently large to be representative. Then it can be reasonably hypothesized that the average stress is equal to the constant boundary traction stress: $\left\langle \sigma_{kl}\right\rangle = \sigma^0_{kl} = \sigma^\infty_{kl}$. We hereafter omit the averaging signs for both stresses and strains. For flat cracks where the unit outer normal is constant for each crack, we have:

$$\varepsilon_{ij} = M^0_{ijkl}\sigma_{kl} + \frac{1}{2V}\sum_k \left(\left\langle b_i\right\rangle n_j + \left\langle b_j\right\rangle n_i\right)^{(k)} s^{(k)} \tag{5}$$

It is thus observed that the contribution of a given crack into the overall strain is proportional to the product of crack surface with COD. It is Hill (1963) who firstly proposed this type of expression for a more general case of arbitrary cavities. Hill (1963) expressed the cavity's contribution into the overall strain as an integral over the cavity surface:

$$\Delta\varepsilon_{ij} = \frac{1}{2V_S}\int_S \left(u_i n_j + u_j n_i\right) ds \tag{5}$$

It reduces to the second term of the (5) when the cavities shrink to cracks. The elastic potential of a cracked solid can be obtained by:

$$\begin{aligned}P\left(\sigma_{ij}\right) &= \frac{1}{2}\sigma_{ij}M_{ijkl}\sigma_{kl} = \frac{1}{2}\sigma_{ij}M^0_{ijkl}\sigma_{kl} + \frac{1}{2V}\sum_k \left(n_i\sigma_{ij}\left\langle b_j\right\rangle\right)^{(k)} s^{(k)} \\ &= P_0\left(\sigma_{ij}\right) + \Delta P\end{aligned} \tag{6}$$

where $P_0\left(\sigma_{ij}\right)$ is the potential of a matrix material without cracks, which has the following the expression for an isotropic (no crack) material with Young's modulus E_0 and Poisson's ratio ν_0:

$$P_0\left(\sigma_{ij}\right)=\frac{1+\nu_0}{2E_0}\sigma_{ij}\sigma_{ij}-\frac{\nu_0}{2E_0}\sigma_{kk}^2 \tag{7}$$

In a 2D case, E_0 is replaced with E_0' where $E_0'=E_0$ for the plane stress case and $E_0'=E_0\big/\left(1-\nu_0^2\right)$ for the plane strain case. ΔP in (6) is the perturbation of potential induced by the introduction of cracks into the material body, which is crucial in determining the effective moduli of the cracked rocks. To this end, Kachanov (1992) introduced a second rank tensor $\mathbf{B}$ to relate the uniform traction $\mathbf{t}$ applied at the crack surface and the resulting average crack opening displacement:

$$\left\langle b_j\right\rangle=t_iB_{ij} \tag{8}$$

which he called *crack compliance tensor*. For an infinite material body, $\mathbf{B}$ demons on crack size, shape and the elastic properties of the matrix and in the case of anisotropic matrix, on the orientation of the crack with respect to the anisotropy axes of the matrix as well. In the case of an single crack in an infinite body with stress at infinity, Eq.(8) can be rewritten as

$$\left\langle b_j\right\rangle=n_l\sigma_{il}B_{ij} \tag{9}$$

B_{ij} is a symmetric tensor. In a local coordinate system $\left(n_n,n_t\right)$ where t is the in plane tangential direction of the crack. The diagonal components COD are denoted by B_n and B_t, respectively. The off-diagonal components characterise coupling of the modes that is relevant for anisotropic matrix and for non-circular cracks in isotropic matrix.

For non-interacting cracks randomly distributed in an infinite isotropic solid, the compliance is linear in crack density parameter, and the normal and shear modes are uncoupled. In the 2D case, $\mathbf{B}$ is proportional to a unit tensor:

$$B_{ij}=\frac{\pi r}{E_0'}\delta_{ij} \tag{10}$$

where r is the half-length of the crack. In this case, the compliances in the normal and shear directions are equal to each other. The perturbed potential ΔP in (6) is now:

$$\Delta P=\frac{\pi}{E_0'}\sigma_{ij}\sigma_{jk}\alpha_{ik} \tag{11}$$

where $\boldsymbol{\alpha}$ is defined in Eq.(2). The elastic potential density P now is a function of σ_{ij} and α_{ij} and ΔP a simultaneous invariant of σ_{ij} (quadratic) and α_{ij} (linear), which ensures the isotropy of the material.

For a 3D case, if the crack is a circular one with radius of r, the compliance tensor is given by Kachanov (1992) by:

$$\mathbf{B}=\frac{32\left(1-\nu_0^2\right)r}{3\pi\left(2-\nu_0\right)E}\left(\mathbf{I}-\frac{\nu_0}{2}\mathbf{nn}\right) \tag{12}$$

where $\mathbf{I}$ is the second-order identity tensor. The perturbation of potential ΔQ in (6) is:

$$\Delta P=\frac{16\left(1-\upsilon_0^2\right)}{3\left(2-\upsilon_0\right)E_0}\left\{(\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}):\boldsymbol{\alpha}-\frac{\upsilon_0}{2}\boldsymbol{\sigma}:\frac{1}{V}\sum_k\left(r^3\mathbf{nnnn}\right)^{(k)}:\boldsymbol{\sigma}\right\} \tag{13}$$

Substitute Eq.(2) into (13) we have:

$$\Delta P=\frac{1}{V}\frac{16\left(1-\upsilon_0^2\right)}{3\left(2-\upsilon_0\right)E_0}\left\{\sum_m\left[r^3\left(\sigma_{ij}\sigma_{jk}n_jn_k-\frac{\upsilon_0}{2}\sigma_{ij}\sigma_{kl}n_in_jn_kn_l\right)\right]^{(m)}\right\} \tag{14}$$

If the crack is of elliptical shape, Kachanov (1992) constructed $\mathbf{B}$ by using Eshelby's ellipsoidal inclusion approach (see also Budiansky and O'Connell, 1976).

By using the result in Eq.(13) or (14), the effective moduli of a cracked rocks can now be estimated. For a rock with randomly distributed cracks, for example:

$$\boldsymbol{\alpha}=\frac{\rho}{3}\mathbf{I} \tag{16}$$

It follows from Eq.(14) that

$$\Delta P=\frac{16\left(1-\upsilon_0^2\right)\rho}{9\left(2-\upsilon_0\right)E_0}\left\{\left(1-\frac{\upsilon_0}{5}\right)\sigma_{ij}\sigma_{ji}-\frac{\upsilon_0}{10}\sigma_{kk}^2\right\} \tag{17}$$

And the corresponding effective moduli can be readily obtained as:

$$\frac{E}{E_0}=\left[1+\frac{16\left(1-\upsilon_0^2\right)\left(10-3\upsilon_0\right)\rho}{45\left(2-\upsilon_0\right)}\right]^{-1},\quad \frac{G}{G_0}=\left[1+\frac{32\left(1-\upsilon_0\right)\left(5-\upsilon_0\right)\rho}{45\left(2-\upsilon_0\right)}\right]^{-1} \tag{18}$$

For a rock with parallel cracks, say the normal of the crack is to the x_1 direction:

$$\boldsymbol{\alpha}=\rho\mathbf{a}_1\mathbf{a}_1 \tag{19}$$

where $\mathbf{a}_1$ is the unit base vector in the x_1 direction. Eq.(14) now has the following expression

$$\Delta P=\frac{16\left(1-\upsilon_0^2\right)\rho}{3\left(2-\upsilon_0\right)E_0}\left\{\sigma_{1j}\sigma_{j1}-\frac{\upsilon_0}{2}\sigma_{11}^2\right\} \tag{20}$$

The corresponding effective moduli in this case are:

$$\frac{E}{E_0}=\left[1+\frac{16\left(1-\upsilon_0^2\right)\rho}{3}\right]^{-1},\qquad \frac{G}{G_0}=\left[1+\frac{16\left(1-\upsilon_0\right)\rho}{3\left(2-\upsilon_0\right)}\right]^{-1} \tag{21}$$

3 Self-consistent scheme by asymptotic approximation of Eshelby's tensor

In deriving the self-consistent equations which govern the effective elastic constants of composites, two approaches have been used in the literature. The first is based on energy consideration (Budiansky, 1975) and the second involves a direct averaging of the components of stress and strain in the constituent phases of the body (Hill, 1965). The two methods are totally equivalent to each other. We consider a solid matrix containing a single ellipsoidal void inclusion with the semi-axes being a_1, a_2 and a_3, respectively. The matrix is assumed to be placed in an equilibrated state of uniform elastic strain $\tilde{\boldsymbol{\varepsilon}}^{\infty}$ by external loads. According to Eshelby theory (Eshelby, 1957, 1959), the inclusion also reaches a state of strain $\tilde{\boldsymbol{\varepsilon}}^{\mathbf{i}}$ which is uniform inside the void, provided $\tilde{\boldsymbol{\varepsilon}}^{\infty}$ is uniform. Eshelby termed this strain as eigenstrain. The relation between the two strains is given by:

$$\tilde{\boldsymbol{\varepsilon}}^{\mathbf{i}} = \left[\tilde{\mathbf{I}} - \tilde{\mathbf{S}}\right]^{-1} : \tilde{\boldsymbol{\varepsilon}}^{\infty} \tag{22}$$

where $\tilde{\mathbf{I}}$ is the fourth-order identity tensor and $\tilde{\mathbf{S}}$ is the Eshelby's tensor, which depends on the aspect ratios of the ellipsoid and on the Poisson ratio ν of the matrix. We assume the semi-axes have the following order of magnitude: $a_1 \geq a_2 \geq a_3 \geq 0$, and define the following two aspect ratio for the ellipsoidal inclusion:

$$e_1 = \frac{a_2}{a_1}, \; e_2 = \frac{a_3}{a_2} \tag{23}$$

In Voigt's notation, the exact mathematical definition of the Eshelby tensor for this geometrical configuration can be found as follows (see, Mura, 1982):

$$\tilde{\mathbf{S}} = \begin{bmatrix} \mathbf{X} & \mathbf{0} \\ \mathbf{0} & \mathbf{Z} \end{bmatrix} \tag{24}$$

Where the submatrices $\mathbf{X}$ and $\mathbf{Z}$ have the following expressions:

$$\mathbf{X} = \frac{1}{8\pi(1-\nu)} \begin{bmatrix} 3I_{11} + (1-2\nu)I_1 & e_1^2 I_{12} - (1-2\nu)I_1 & e_1^2 e_2^2 I_{13} - (1-2\nu)I_1 \\ I_{12} - (1-2\nu)I_2 & 3e_1^2 I_{22} + (1-2\nu)I_2 & e_1^2 e_2^2 I_{23} - (1-2\nu)I_2 \\ I_{13} - (1-2\nu)I_3 & e_1^2 I_{23} - (1-2\nu)I_3 & 3e_1^2 e_2^2 I_{33} + (1-2\nu)I_3 \end{bmatrix}$$

$$\mathbf{Z} = \frac{1}{8\pi(1-\nu)} \begin{bmatrix} Z_{11} & 0 & 0 \\ 0 & Z_{22} & 0 \\ 0 & 0 & Z_{33} \end{bmatrix}$$

And

$$I_1 = \frac{4\pi e_1^2 e_2 \left[\mathbb{F}(p,q) - \mathbb{E}(p,q)\right]}{\left(1-e_1^2\right)\sqrt{1-e_1^2 e_2^2}} \quad , \quad I_2 = \frac{4\pi e_2\left(1-e_1^2 e_2^2\right)\mathbb{E}(p,q)}{\left(1-e_1^2\right)\left(1-e_2^2\right)\sqrt{1-e_1^2 e_2^2}} - \frac{4\pi e_1^2 e_2 \mathbb{F}(p,q)}{\left(1-e_1^2\right)\sqrt{1-e_1^2 e_2^2}} - \frac{4\pi e_2^2}{\left(1-e_2^2\right)} \quad ,$$

$$I_3 = \frac{4\pi}{\left(1-e_2^2\right)} - \frac{4\pi e_2 \mathbb{E}(p,q)}{\left(1-e_2^2\right)\sqrt{1-e_1^2 e_2^2}}, \qquad p = \arcsin\sqrt{1-e_1^2 e_2^2}, \; q = \sqrt{\frac{1-e_1^2}{1-e_1^2 e_2^2}}$$

$$Z_{11} = \left(1+e_1^2\right)I_{12} + (1-2\nu)(I_1 + I_2), \; Z_{22} = e_1^2\left(1+e_2^2\right)I_{23} + (1-2\nu)(I_2 + I_3),$$

$$Z_{33} = \left(1+e_1^2 e_2^2\right)I_{13} + (1-2\nu)(I_1 + I_3)$$

$$\mathbb{F}(p,q) = \int_0^p \frac{d\alpha}{\sqrt{1-q^2\sin^2\alpha}}, \; \mathbb{E}(p,q) = \int_0^p \sqrt{1-q^2\sin^2\alpha}\, d\alpha$$

$$I_{11} = \frac{I_1\left(e_1^4 e_2^2 - 2e_1^2 e_2^2 - 2e_1^2 + 3\right) + I_2 e_1^2\left(e_1^2 e_2^2 - 1\right) + I_3 e_1^2 e_2^2\left(e_1^2 - 1\right)}{3\left(1-e_1^2\right)\left(1-e_1^2 e_2^2\right)}$$

$$I_{22} = \frac{I_1\left(1-e_2^2\right) + I_2\left(2e_1^2 e_2^2 - 3e_1^2 + 2 - e_2^2\right) + I_3 e_2^2\left(e_1^2 - 1\right)}{3e_1^2\left(1-e_1^2\right)\left(1-e_2^2\right)}$$

$$I_{33}=\frac{I_1\left(1-e_2^2\right)+I_2\left(1-e_1^2e_2^2\right)+I_3\left(1-2e_1^2e_2^2+3e_1^2e_2^4-2e_2^2\right)}{3e_1^2e_2^2\left(1-e_2^2\right)\left(1-e_1^2e_2^2\right)}$$

$$I_{12}=\frac{I_2-I_1}{1-e_1^2},\ I_{13}=\frac{I_3-I_1}{1-e_1^2e_2^2},\ I_{23}=\frac{I_3-I_2}{e_1^2\left(1-e_2^2\right)}$$

The general Eshelby’s tensor in this case is actually a function of υ, e_1 and e_2:

$$\tilde{\mathbf{S}}=\tilde{\mathbf{S}}\left(\upsilon, e_1, e_2\right) \tag{25}$$

When a crack of any type is concerned (such that the aspect ratio $e_2 \to 0$), the $\tilde{\mathbf{S}}$ tensor can be approximated by its Taylor series with respect to e_2 about $e_2=0$:

$$\tilde{\mathbf{S}}=\tilde{\mathbf{S}}\Big|_{e_2=0}+\frac{\partial \tilde{\mathbf{S}}}{\partial e_2}\bigg|_{e_2=0} e_2+\mathbf{O}\left(e_2^2\right) \tag{26}$$

If we write

$$\mathbf{S}=\tilde{\mathbf{S}}\Big|_{e_2=0} \text{ and } \mathbf{M}=\frac{\partial \tilde{\mathbf{S}}}{\partial e_2}\bigg|_{e_2=0} \tag{27}$$

Then we have:

$$\mathbf{S}=\mathbf{S}\left(\upsilon, e_1\right),\ \ \mathbf{M}=\mathbf{M}\left(\upsilon, e_1\right) \tag{28}$$

Therefore, if we neglect the higher-order terms other than the first two terms in the right-hand side of (8), the following asymptotic form can be regarded as a good approximation of the Eshelby’s tensor for a cracks:

$$\tilde{\mathbf{S}} \to \mathbf{S}+\mathbf{M} e_2 \tag{29}$$

If penny-shaped cracks are treated such that $a_1=a_2=r$ and $e_1=1$ is very small, we further have (see, Yang et al., 2001):

$$\mathbf{S}^p=\mathbf{S}^p\left(\upsilon\right) \text{ and } \mathbf{M}^p=\mathbf{M}^p\left(\upsilon\right) \tag{30}$$

where the superscript p denotes penney-shaped cracks. From the definition of Eshelby’s tensor in Eq.(24) we know the following limits for $\mathbf{S}^p$ when $e_2 \to 0$

$$S_{2323}^p=S_{2332}^p=S_{3223}^p=S_{3232}^p=0.5,\ \ S_{1313}^p=S_{1331}^p=S_{3113}^p=S_{3131}^p=0.5,$$
$$S_{3311}^p=S_{3322}^p=\upsilon/\left(1-\upsilon\right),\ S_{3333}^p=1 \tag{31}$$

with all other elements vanishing.

The above expressions are defined in the local coordinate system where the principal axes of the crack coincide with the coordinate directions. Suppose the directional cosines of this local coordinate system with respect to a global system are $n_i^{(1)}, n_i^{(2)}, n_i^{(3)}$, respectively. If in the local system the Eshebly’s tensor is $\mathbf{S}'$, in the global coordinate system the tensor $\mathbf{S}$ can be determined by the following transformation:

$$S_{ijkl}=n_i^{(m)} n_j^{(n)} n_k^{(p)} n_l^{(q)} S'_{mnpq} \tag{32}$$

note that $n_i^{(1)} n_j^{(1)}+n_i^{(2)} n_j^{(2)}+n_i^{(3)} n_j^{(3)}=\delta_{ij}$.

We now restrict our discussion to the case of penny-shape crack only. In this case, the global tensor $\mathbf{S}$ can be written in the following form:

$$S_{ijkl} = \frac{1}{2}\left(\delta_{ik}n_j n_l + \delta_{il}n_j n_k + \delta_{jk}n_i n_l + \delta_{jl}n_i n_k - 4n_i n_j n_k n_l\right) + \frac{\upsilon}{1-\upsilon}n_i n_j\left(\delta_{kl} - n_k n_l\right) + n_i n_j n_k n_l \qquad (33)$$

It is readily to verify that (33) reduces to (31) when $\mathbf{n} = \{0, 0, 1\}$. $\mathbf{S}$ can be further decomposed into two parts:

$$\mathbf{S} = \mathbf{S}^n + \mathbf{S}^t \qquad (34)$$

where

$$\mathbf{S}^n = \frac{\upsilon}{1-\upsilon}\mathbf{nnI} + \frac{1-2\upsilon}{1-\upsilon}\mathbf{nnnn}\,,\quad S^t_{ijkl} = \frac{1}{2}\left(n_i n_k \delta_{jl} + n_i n_l \delta_{jk} + n_j n_k \delta_{il} + n_j n_l \delta_{ik} - 4n_i n_j n_k n_l\right) \qquad (35)$$

Analogously, the global tensor $\mathbf{M}$ transformed from a local one has the following expression:

$$\begin{aligned} M_{ijkl} &= \frac{\pi(\upsilon-2)}{8(1-\upsilon)}\left(\delta_{ik}n_j n_l + \delta_{il}n_j n_k + \delta_{jk}n_i n_l + \delta_{jl}n_i n_k - 4n_i n_j n_k n_l\right) \\ &\quad + \frac{\pi(4\upsilon+1)}{8(1-\upsilon)}n_i n_j\left(\delta_{kl} - n_k n_l\right) + \frac{\pi(2\upsilon-1)}{8(1-\upsilon)}n_k n_l\left(\delta_{ij} - n_i n_j\right) + \frac{\pi(2\upsilon-1)}{4(1-\upsilon)}n_i n_j n_k n_l \end{aligned} \qquad (36)$$

The energy perturbation produced by a single isolated crack in an infinite medium, as stated by Budiansky and O'Connell (1976), is only influenced by the resolved normal stress σ and τ normal and tangential to the plane of the crack and must be a quadratic function of these stresses. Yang et al. (2001) presented a mathematical derivation of the expression for this energy perturbation based on the asymptotic approximation method. For an isotropic media, an asymptotic expression for the eigenstrain is firstly found in terms of $\mathbf{S}^n$, $\mathbf{S}^t$ and the aspect ratio e_2:

$$\tilde{\boldsymbol{\varepsilon}}^{\mathbf{i}} \to \frac{1}{e_2}\left(\frac{4(1-\upsilon)^2}{\pi(1-2\upsilon)}\mathbf{S}_n + \frac{4(1-\upsilon)}{\pi(2-\upsilon)}\mathbf{S}_n\right) : \tilde{\boldsymbol{\varepsilon}}^{\infty} \qquad (37)$$

Note that in deriving (37), Eq.(36) has been used. Then the energy perturbation is obtained by:

$$\begin{aligned} \Delta Q &= \frac{V}{2}\boldsymbol{\sigma}^{\infty} : \tilde{\boldsymbol{\varepsilon}}^{\mathbf{i}} = \lim_{e_2 \to 0}\frac{1}{2}\frac{4\pi}{3}r^3 e_2 \boldsymbol{\sigma}^{\infty} : \left(\frac{1}{e_2}\left(\frac{4(1-\upsilon)^2}{\pi(1-2\upsilon)}\mathbf{S}_n + \frac{4(1-\upsilon)}{\pi(2-\upsilon)}\mathbf{S}_n\right) : \tilde{\boldsymbol{\varepsilon}}^{\infty}\right) \\ &= \frac{1}{2}\frac{4\pi}{3}r^3 \boldsymbol{\sigma}^{\infty} : \left(\frac{4(1-\upsilon)^2}{\pi(1-2\upsilon)}\mathbf{S}_n + \frac{4(1-\upsilon)}{\pi(2-\upsilon)}\mathbf{S}_n\right) : \tilde{\boldsymbol{\varepsilon}}^{\infty} \\ &= \frac{1}{2}\frac{4\pi}{3}r^3 \boldsymbol{\sigma}^{\infty} : \left(\frac{4(1-\upsilon)^2}{\pi(1-2\upsilon)}\mathbf{S}_n + \frac{4(1-\upsilon)}{\pi(2-\upsilon)}\mathbf{S}_n\right) : \left(\mathbf{C} : \boldsymbol{\sigma}^{\infty}\right) \end{aligned} \qquad (38)$$

where $\mathbf{C}$ is the compliance tensor of the virgin material, $\mathbf{C} = -\frac{\upsilon}{E}\mathbf{I}^{(2)}\mathbf{I}^{(2)} + \frac{1+\upsilon}{E}\mathbf{I}^{(4)}$. It is easy to find that:

$$\mathbf{S}^n : \mathbf{C} = \frac{(1+\upsilon)(1-2\upsilon)}{E(1-\upsilon)}\mathbf{N}\,, \qquad \mathbf{S}^t : \mathbf{C} = \frac{2(1+\upsilon)}{E}\mathbf{T} \qquad (39)$$

where the tensor $\mathbf{N}$ and $\mathbf{T}$ are associated with the normal and tangential components of a stress tensor in the direction of : $\mathbf{n}$

$$\mathbf{N} = \mathbf{nnnn}\ ,\ \mathbf{T} = \frac{1}{2}\mathbf{S}^t \tag{40}$$

Eq.(38) can thus be reformulated as:

$$\Delta Q = \frac{r^3}{E}\left[\sigma_0^2 f(\upsilon) + \tau_0^2 g(\upsilon)\right] \tag{41}$$

where

$$\sigma_0^2 = \boldsymbol{\sigma}^\infty : \mathbf{N} : \boldsymbol{\sigma}^\infty\ ,\ \ \tau_0^2 = \boldsymbol{\sigma}^\infty : \mathbf{T} : \boldsymbol{\sigma}^\infty\ ,\ \ f(\upsilon) = \frac{8}{3}\left(1-\upsilon^2\right),\ g(\upsilon) = \frac{16\left(1-\upsilon^2\right)}{3\left(2-\upsilon\right)} \tag{42}$$

Eq.(41) recovers the exact same form as developed by Budiansky and O'Connell (1976). Now we rewrite Eq.(41) :

$$\begin{aligned}
\Delta Q &= \frac{8r^3\left(1-\upsilon^2\right)}{3E}\left[\sigma_{ij} n_i n_j n_k n_l \sigma_{kl} + \frac{1}{2\left(2-\upsilon\right)}\sigma_{ij}\left(n_i n_k \delta_{jl} + n_i n_l \delta_{jk} + n_j n_k \delta_{il} + n_j n_l \delta_{ik} - 4 n_i n_j n_k n_l\right)\sigma_{kl}\right] \\
&= \frac{8r^3\left(1-\upsilon^2\right)}{3E}\left[\sigma_{ij}\sigma_{kl} n_i n_j n_k n_l + \frac{1}{2\left(2-\upsilon\right)}\left(4\sigma_{ij}\sigma_{jk} n_i n_k - 4\sigma_{ij}\sigma_{kl} n_i n_j n_k n_l\right)\right] \\
&= \frac{8r^3\left(1-\upsilon^2\right)}{3E}\left[\sigma_{ij}\sigma_{kl} n_i n_j n_k n_l + \frac{2}{\left(2-\upsilon\right)}\left(\sigma_{ij}\sigma_{jk} n_i n_k - \sigma_{ij}\sigma_{kl} n_i n_j n_k n_l\right)\right] \\
&= \frac{16r^3\left(1-\upsilon^2\right)}{3\left(2-\upsilon\right)E}\left[\sigma_{ij}\sigma_{jk} n_i n_k - \frac{\upsilon}{2}\sigma_{ij}\sigma_{kl} n_i n_j n_k n_l\right]
\end{aligned} \tag{43}$$

Eq.(37) represents the energy perturbation contributed by a single crack. If it is assumed the cracks are non-interacting such that the energy perturbation for all cracks can be superposed, we are able to sum up Eq.(43) for all cracks in the material body and then divide it by the material volume V , which leads to the following potential density: $\sum \Delta Q / V$. It is readily to observe that this result is identical with the one in Eq.(14) which was derived from a non-interacting method. However, a key point the self-consistent method is based on is that a crack is placed into a matrix with the effective elastic moduli, not the virgin ones. For randomly distributed cracks, the elastic potential is thus now a function of the effective Young's modulus $\bar{E}$ and effective Poisson's ratio $\bar{\upsilon}$, both of which are dependent on the crack density ρ . As a result we now have:

$$\frac{\Delta Q}{V} = \frac{16\left(1-\left(\upsilon(\rho)\right)^2\right)\rho}{9\left(2-\upsilon(\rho)\right)E(\rho)}\left\{\left(1-\frac{\upsilon(\rho)}{5}\right)\sigma_{ij}\sigma_{ji} - \frac{\upsilon(\rho)}{10}\sigma_{kk}^2\right\} \tag{43}$$

The total elastic potential is then:

$$\Delta P = \left(\frac{1+\nu_0}{2E_0} + \frac{16\left(1-\upsilon^2\right)\left(5-\upsilon\right)\rho}{45\left(2-\upsilon\right)E}\right)\sigma_{ij}\sigma_{ij} - \left(\frac{\nu_0}{2E_0} + \frac{8\left(1-\upsilon^2\right)\upsilon\rho}{45\left(2-\upsilon\right)E}\right)\sigma_{kk}^2 \tag{44}$$

Therefore we have:

$$
\begin{cases}
\dfrac{1}{E_0}+\dfrac{16\left(1-\upsilon^2\right)\left(5-\upsilon\right)\left(10-3\upsilon\right)\rho}{45\left(2-\upsilon\right)E}=\dfrac{1}{E} \\
\dfrac{1+\nu_0}{E_0}+\dfrac{32\left(1-\upsilon^2\right)\left(5-\upsilon\right)\rho}{45\left(2-\upsilon\right)E}=\dfrac{\left(1+\upsilon\right)}{E}
\end{cases}
\tag{45}
$$

Solving (45) for $\overline{E}$ and $\overline{\upsilon}$ we can then obtain the effective moduli of the cracked rocks by self-consistent scheme, which are essentially softer than the non-interacting results at larger crack densities. It is assumed the inclusion of cracks in the matrix does not change the Poisson's ratio too much such that one can safely assume that $\overline{\upsilon}=\upsilon$. In this case, the self-consistent results lead to the following solutions to the elastic moduli:

$$
\frac{E}{E_0}=1-\frac{16\left(1-\upsilon_0^2\right)\left(5-\upsilon_0\right)\left(10-3\upsilon_0\right)\rho}{45\left(2-\upsilon_0\right)}, \quad \frac{G}{G_0}=1-\frac{32\left(1-\upsilon_0\right)\left(5-\upsilon_0\right)\rho}{45\left(2-\upsilon_0\right)} \tag{46}
$$

It is interesting to point out that, based on an assumption that the Poisson's ratio does not change significantly by the cracks, Berryman et al. (2002) employed a differential scheme and derived the following approximate analytical solutions for a randomly distributed cracked rock:

$$
\frac{E}{E_0}=\left(1-\rho\right)^{\frac{4\left(1-\upsilon_0^2\right)}{3\pi e_2\left(1-2\upsilon_0\right)}}, \quad \frac{\overline{G}}{G}=\left(1-\rho\right)^{\frac{1}{5}\left[1+\frac{8\left(1-\upsilon_0\right)\left(5-\upsilon_0\right)}{3\pi e_2\left(2-\upsilon_0\right)}\right]} \tag{47}
$$

where e_2 is the aspect ratio defined in Eq.(23). Eq.(47) takes into account the effect of aspect ratio on the effective moduli.

Figure 1 and Figure 2 present the effective moduli predicted by Eqs.(18), (46) and (47) for a rock with randomly distributed cracks in 3D case. Four values of e_2 are used for Eq.(47). As can be seen, Eq.(47) predicts a effective Young's modulus whose value is between that by Eq.(18) and Eq.(46). The larger e_2 is, the higher E is. For effective shear modulus, however, Eq.(47) gives magnitude between those by Eq.(18) and (46) only when e_2 is large. When e_2 is as small as 0.1, Eq.(47) gives the smallest G of the three.

4 Discussion and conclusion

We have used a non-interacting approach based on crack compliance tensor and a self-consistent approach based on an asymptotic approximation of the Eshelby's tensor to investigate the problem of effective elastic properties of cracked rocks. It is shown the asymptotic method can be used to estimate the effective moduli and can be easily degenerated to the non-interacting approach when the energy perturbation is assumed to be superposable. The non-interacting method is generally known as good approximation when the crack density is low. The same can be said for the self-consistent scheme. When crack density is high, a well-know problem occurs for SCS. For porous media with spherical voids the SCS predicts that the effective elastic moduli diminish linearly with void volume fraction until they vanish at 50%, which is in disagreement with experiemental results. Similar problems are encountered in the SCS results for penny-shaped cracks. The effective Young's modulus diminishes linearly with crack density parameter and the effective shear modulus nearly so, both moduli abruptly vanishing for a value of crack density parameter at 9/16. Similar behaviour can be observed by the approximation in Eq.(46) in Figure 1 and 2. This is not reasonable since the moduli should vanish asymptotically with increasing crack density. Linear variation with crack density parameter is restricted to small crack density when the cracks do not interact and is not therefore necessarily valid for large density. There are still some issued needing to be resolved on the study of cracked rocks. In a recent comment, Kachanov (2007) pointed out that in deriving the effective elastic properties of cracked solids, the importance of crack shapes and influence of fluid filled in the cracks and the definition on the crack density parameter in relation with the approximate schemes should be taken into account. Steady effort is needed towards these areas on the study of cracked rocks.

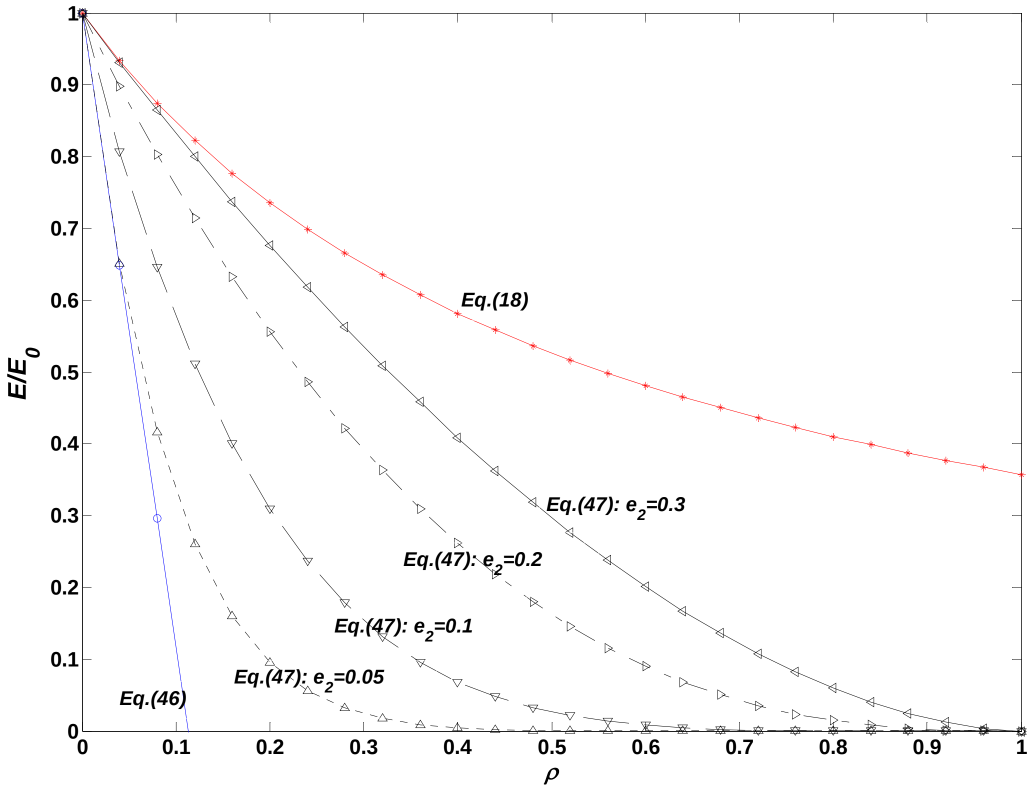

Figure 1 Effective Young's modulus for a rock with randomly distributed cracks in 3D case.

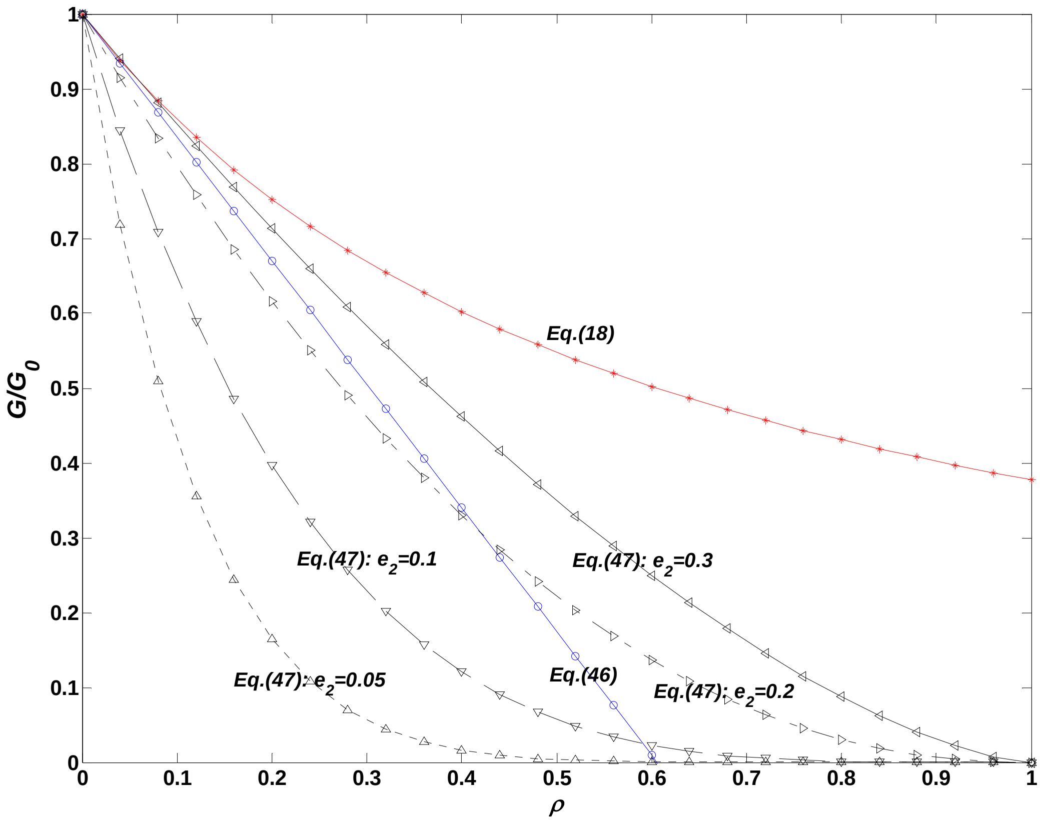

Figure 2 Effective shear modulus for a rock with randomly distributed cracks in 3D case.

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